

Perturbation determinants and trace formulas for singular perturbations

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Abstract

We use the boundary triplet approach to extend the classical concept of perturbation determinants to a more general setup. In particular, we examine the concept of perturbation determinants to pairs of proper extensions of closed symmetric operators. For an ordered pair of extensions we express the perturbation determinant in terms of the abstract Weyl function and the corresponding boundary operators. A crucial role in our approach plays so-called almost solvable extensions. We obtain trace formulas for pairs of self-adjoint, dissipative and other pairs of extensions and express the spectral shift function in terms of the abstract Weyl function and the characteristic function of almost solvable extensions.

We emphasize that for *pairs of dissipative extensions our results are new even for the case of additive perturbations*. In this case we improve and complete some classical results of M.G. Krein for pairs of self-adjoint and dissipative operators. We apply the main results to ordinary differential operators and to elliptic operators as well.

Keywords: symmetric operators, proper extensions, almost solvable extensions, perturbation determinants, trace formulas, spectral shift functions

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1 Introduction

The perturbation determinant (perturbation determinant) was introduced and used by Krein in [41, 42, 43, 44, 45]. Independently from Krein it was also introduced by Kuroda in [47]. The perturbation determinant is an important tool to study the spectral shift function for pairs of self-adjoint operators [7, 8, 41, 44, 45] as well as for non-selfadjoint operators [46]. Moreover, it was also used to analyze certain other properties of non-selfadjoint operators as the completeness of the root vectors etc, cf. [42, 43]. Recently, perturbation determinants in application to Schrödinger operators were studied in [27, 24, 28, 29, 23, 25, 30, 31]. The properties of perturbation determinants are summarized in [10, 32, 65].

A perturbation determinant relates a scalar-valued holomorphic function to an ordered pair of closed linear operators $\{H', H\}$ defined on some Hilbert space \mathfrak{H} . In the simplest case if H' and H are bounded operators such $V := H' - H$ is a trace class operator the perturbation determinant is defined by

$$\Delta_{H'/H}(z) := \det(I + V(H - z)^{-1}), \quad z \in \rho(H). \quad (1.1)$$

Obviously, the definition extends to unbounded operators H' and H if $\text{dom}(H') = \text{dom}(H)$ and $V := \overline{H' - H}$ is a trace class operator provided the resolvent set $\rho(H)$ is not empty. In [44, 46] Krein has introduced a class \mathfrak{D} of pairs of densely defined closed operators $\{H', H\}$ given by

- (i) $\rho(H') \cap \rho(H)$ is not empty,
- (ii) $\text{dom}(H') = \text{dom}(H)$,
- (iii) $(H' - H)(H - z)^{-1} \in \mathfrak{S}_1(\mathfrak{H})$ for $z \in \rho(H)$.

to which the definition (1.1) can be extended. Pairs $\{H', H\}$ of densely defined closed operators satisfying the conditions (i) and (ii) are called regular in the following. Therefore, Krein's theory of perturbation determinant makes sense for regular pairs of operators.

However, non-regular pairs $\{\tilde{A}', \tilde{A}\}$ naturally appear in consideration of boundary value problems for differential operators. For instance, any pair $\{\tilde{A}', \tilde{A}\}$ of different proper extensions of a densely defined closed symmetric operator A is not regular: $\text{dom}(\tilde{A}') = \text{dom}(\tilde{A})$ if and only if $\tilde{A}' = \tilde{A}$. In the sequel a pair $\{\tilde{A}', \tilde{A}\}$ of closed operators is called singular if there is a densely defined closed symmetric operator A such that \tilde{A}' and \tilde{A} are proper extensions of A . The aim of the paper is to extend Krein's theory of perturbation determinants from regular to singular pairs.

To do this there are several approaches. For instance (see [10], [65]) for a pair $\{H', H\}$ of densely defined closed operators with the trace class resolvent difference the following concept of generalized perturbation determinant was proposed

$$\tilde{\Delta}_{H'/H}(z, \xi) := \det((H' - z)(H' - \xi)^{-1}(H - \xi)(H - z)^{-1}), \quad (1.2)$$

$z \in \rho(H)$ and $\xi \in \rho(H')$. However, this definition has few drawbacks. One of them is that it cannot be applied to boundary value problems.

In the present paper we propose a new concept of generalized perturbation determinants by applying the technique of boundary triplets and the corresponding Weyl functions (see Section 2 for precise definitions). This new approach to extension theory of symmetric operators has been appeared and elaborated during the last three decades (see [14, 15, 16, 33, 49] and references therein).

Recall that a triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where \mathcal{H} is an auxiliary Hilbert space and $\Gamma_0, \Gamma_1 : \text{dom}(A^*) \rightarrow \mathcal{H}$ are linear mappings, is called a boundary triplet for the adjoint A^* of a symmetric operator A if the "abstract Green's identity"

$$(A^*f, g) - (f, A^*g) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \quad f, g \in \text{dom}(A^*), \quad (1.3)$$

holds and the mapping $\Gamma := (\Gamma_0, \Gamma_1)^\top : \text{dom}(A^*) \rightarrow \mathcal{H} \oplus \mathcal{H}$ is surjective.

A boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* always exists whenever $n_+(A) = n_-(A)$, though it is not unique. Its role in extension theory is similar to that of a coordinate system in analytic geometry. It leads to a natural parametrization of the set Ext_A of proper extensions \tilde{A} of A ($A \subset \tilde{A} \subset A^*$) by means of the set $\tilde{\mathcal{C}}(\mathcal{H})$ of linear relations (multi-valued operators) in \mathcal{H} , see [33] and [16] for detailed treatments. In this paper we consider only the case of boundary relation Θ being the graph $\text{gr}(B)$ of a closed linear operator B in \mathcal{H} , i.e. assume that the extension \tilde{A} is given by

$$A_B := A^* \upharpoonright \ker(\Gamma_1 - B\Gamma_0). \quad (1.4)$$

In this case the extension parameter is often called the boundary operator.

The main analytical tool in this approach is the abstract Weyl function $M(\cdot)$ which was introduced and studied in [16]. This Weyl function plays a similar role in the theory of boundary triplets as the classical Weyl-Titchmarsh function does in the theory of Sturm-Liouville operators (see [12, 16, 49, 51]).

This approach to the perturbation determinant for singular pairs allows to express it in terms of the Weyl function [15, 16, 19] and the corresponding boundary operator.

Definition 1.1. Let A be a densely defined closed symmetric operator in \mathfrak{H} , let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* and $M(\cdot)$ the corresponding Weyl function. We say that the ordered pair $\{\tilde{A}', \tilde{A}\}$ of proper extensions of A belongs to the class \mathfrak{D}^Π if \tilde{A}' and \tilde{A} admit representations (1.4) with closed boundary operators B' and B , respectively, and the following conditions are valid

- (i) the set $\{z \in \rho(A_0) : 0 \in \rho(B - M(z))\}$ is not empty,
- (ii) $\text{dom}(B') = \text{dom}(B)$,
- (iii) $(B' - B)(B - M(z))^{-1} \in \mathfrak{S}_1(\mathfrak{H})$ for $z \in \rho(A_0)$ obeying $0 \in \rho(B - M(z))$

where A_0 is a self-adjoint extension of A given by $A_0 := A^* \upharpoonright \ker(\Gamma_0)$. If $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$, then the scalar-valued function

$$\Delta_{\tilde{A}'/\tilde{A}}^\Pi(z) := \det(I_{\mathcal{H}} + (B' - B)(B - M(z))^{-1}) \quad (1.5)$$

defined for all those $z \in \rho(A_0)$ satisfying $0 \in \rho(B - M(z))$ is called the perturbation determinant of the pair $\{\tilde{A}', \tilde{A}\}$ with respect to Π .

In the following we verify that the so defined perturbation determinant $\Delta_{\tilde{A}'/\tilde{A}}^\Pi(\cdot)$ fulfills all standard properties of Krein's perturbation determinant listed in Appendix B, see also [65, Chapter 8.1.1].

The definition of the perturbation determinant for proper extensions depends on the chosen boundary triplet Π . However, it turns out that if $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$ and $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^{\Pi'}$, then the perturbation determinants $\Delta_{\tilde{A}'/\tilde{A}}^\Pi(z)$ and $\Delta_{\tilde{A}'/\tilde{A}}^{\Pi'}(z)$ differ only by a multiplicative complex constant. Hence, the definition of the perturbation in the sense of Definition 1.1 does not depend so much on Π as it seems to be at first glance.

Further, the natural question arises whether for two extensions \tilde{A}' and \tilde{A} of A satisfying

$$(\tilde{A}' - z)^{-1} - (\tilde{A} - z)^{-1} \in \mathfrak{S}_1(\mathfrak{H}), \quad z \in \rho(\tilde{A}') \cap \rho(\tilde{A}). \quad (1.6)$$

there is always a boundary triplet Π for A^* such that $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$. We show that this holds if \tilde{A} is almost solvable, cf. Section 3.

The most simple expression for the perturbation determinant we obtain if the symmetric operator A has finite deficiency indices $n_+(A) = n_-(A) < \infty$. Namely, in this case, as an immediate consequence of Definition 1.1, we arrive at the following formula for the perturbation determinant

$$\Delta_{\tilde{A}'/\tilde{A}}^\Pi(z) := \frac{\det(B' - M(z))}{\det(B - M(z))}, \quad z \in \rho(\tilde{A}') \cap \rho(\tilde{A}). \quad (1.7)$$

For instance, let $A := A_{\min}$ be a minimal symmetric operator generated in $L^2(\mathbb{R}_+)$ by the Sturm-Liouville differential expression

$$\mathcal{L} = -D^2 + q, \quad q = \bar{q} \in l_{\text{loc}}^1[0, \infty),$$

assuming the limit point case at infinity. Let also $L_j := A_{h_j}, j \in \{1, 2\}$, be a proper extension of A determined by

$$\text{dom}(A_{h_j}) = \{y \in \text{dom}(A^*) : y'(0) = h_j y(0)\}, \quad j \in \{1, 2\}.$$

Then

$$\Delta_{L_2/L_1}(z) = \frac{m(z) - h_2}{m(z) - h_1},$$

where $m(\cdot)$ is the Weyl function corresponding to the Dirichlet extension.

In the case of infinite deficiency indices $n_\pm(A) = \infty$ we consider proper extensions $\tilde{A}', \tilde{A} \in \text{Ext}_A$ satisfying $\rho(\tilde{A}') \cap \rho(\tilde{A}) \neq \emptyset$ and $(A_0 := A^* \upharpoonright \ker(\Gamma_0))$

$$(\tilde{A}' - \zeta)^{-1} - (A_0 - \zeta)^{-1} \in \mathfrak{S}_1(\mathfrak{H}) \quad \text{and} \quad (\tilde{A} - \zeta)^{-1} - (A_0 - \zeta)^{-1} \in \mathfrak{S}_1(\mathfrak{H})$$

for $\zeta \in \rho(\tilde{A}') \cap \rho(\tilde{A}) \cap \mathbb{C}_\pm$. Then we can choose a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* such that $\tilde{A}' = A_{B'}$, $\tilde{A} = A_B$ with $B', B \in \mathcal{C}(\mathcal{H})$ and satisfying $(B' - \mu)^{-1}, (B - \mu)^{-1} \in \mathfrak{S}_1(\mathcal{H})$ for some $\mu \in \rho(B') \cap \rho(B) \cap \mathbb{R}$.

Denoting by $M(\cdot)$ the corresponding Weyl function, we show that there exists a boundary triplet $\tilde{\Pi} = \{\tilde{\mathcal{H}}, \tilde{\Gamma}_0, \tilde{\Gamma}_1\}$ for A^* such that $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^{\tilde{\Pi}}$, $\{\tilde{A}', A_0\} \in \mathfrak{D}^{\tilde{\Pi}}$ and $\{\tilde{A}, A_0\} \in \mathfrak{D}^{\tilde{\Pi}}$, and the perturbation determinant $\Delta_{\tilde{A}/A_0}^{\tilde{\Pi}}(\cdot)$ admits a representation

$$\Delta_{\tilde{A}/A_0}^{\tilde{\Pi}}(z) = \det(I - (\mu - B)^{-1}(\mu - M(z))), \quad z \in \rho(A_0). \quad (1.8)$$

Combining this formula with similar formula for $\Delta_{\tilde{A}'/A_0}^{\tilde{\Pi}}(z)$ and using the chain rule we arrive at the formula for $\Delta_{\tilde{A}'/\tilde{A}}^{\tilde{\Pi}}(\cdot)$

$$\Delta_{\tilde{A}'/\tilde{A}}^{\tilde{\Pi}}(z) = \frac{\det(I - (\mu - B')^{-1}(\mu - M(z)))}{\det(I - (\mu - B)^{-1}(\mu - M(z)))}, \quad z \in \rho(\tilde{A}) \cap \mathbb{C}_{\pm}. \quad (1.9)$$

If $0 \in \rho(B') \cap \rho(B)$, we can put $\mu = 0$ in both formulas (1.8) and (1.9).

Formula (1.9) gives a desired extension of (1.7) to the case $n_{\pm}(A) = \infty$ while it is also useful in the case $n_{\pm}(A) < \infty$ (see Section 7.2).

Formulas (1.8) and (1.9) can be applied to boundary value problems for partial differential equations. For instance, consider the Schrödinger symmetric operator in a domain $\Omega \subset \mathbb{R}^2$ with smooth compact boundary,

$$\mathcal{A} := -\Delta + q(x) = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + q(x), \quad q = \bar{q} \in C^\infty(\overline{\Omega}). \quad (1.10)$$

Consider Robin-type realizations of the expression \mathcal{A} ,

$$\begin{aligned} \hat{A}_{\sigma_j} &:= A_{\max} \upharpoonright \text{dom}(\hat{A}_{\sigma_j}), \\ \text{dom}(\hat{A}_{\sigma_j}) &:= \{f \in H^2(\Omega) : G_1 f = \sigma G_0 f\}, \quad j \in \{1, 2\}, \end{aligned} \quad (1.11)$$

and denote by A_0 the Dirichlet realization of \mathcal{A} given by $\text{dom}(\hat{A}_0) = \{f \in H^2(\Omega) : G_0 f = 0\}$. Here G_0 and G_1 are trace operators, $G_0 u := \gamma_0 u := u|_{\partial\Omega}$ and $G_1 u := \gamma_0(\partial u / \partial \nu)$, $u \in \text{dom}(A_{\max})$. It is known that $\hat{A}_0 = (\hat{A}_0)^*$ and the realization \hat{A}_{σ_j} is closed whenever $\sigma_j \in C^2(\partial\Omega)$ and self-adjoint if σ is real.

Denote by $\hat{\sigma}_j$ the multiplication operator induced by σ_j in $L^2(\partial\Omega)$. Assuming that $0 \in \rho(\hat{A}_{\sigma_1}) \cap \rho(\hat{A}_{\sigma_2}) \cap \rho(\hat{A}_0)$, we indicate a boundary triplet $\tilde{\Pi}$ for A_{\max} such that $\{\hat{A}_{\sigma_j}, \hat{A}_0\} \in \mathfrak{D}^{\tilde{\Pi}}$ and the corresponding perturbation determinants $\Delta_{\hat{A}_{\sigma_j}/A_0}^{\tilde{\Pi}}(\cdot)$ and $\Delta_{\hat{A}_{\sigma_2}/A_{\sigma_1}}^{\tilde{\Pi}}(\cdot)$ are given by

$$\Delta_{\hat{A}_{\sigma_j}/A_0}^{\tilde{\Pi}}(z) = \det_{L^2(\partial\Omega)}(I - (\Lambda_{0,0}(z) - \Lambda_{0,0}(0))(\hat{\sigma}_j - \Lambda_{0,0}(0))^{-1}), \quad j \in \{1, 2\},$$

and

$$\Delta_{\hat{A}_{\sigma_2}/A_{\sigma_1}}^{\tilde{\Pi}}(z) = \frac{\det_{L^2(\partial\Omega)}(I - (\Lambda_{0,0}(z) - \Lambda_{0,0}(0))(\hat{\sigma}_2 - \Lambda_{0,0}(0))^{-1})}{\det_{L^2(\partial\Omega)}(I - (\Lambda_{0,0}(z) - \Lambda_{0,0}(0))(\hat{\sigma}_1 - \Lambda_{0,0}(0))^{-1})},$$

$z \in \rho(\hat{A}_{\sigma_1}) \cap \rho(\hat{A}_{\sigma_2}) \cap \rho(\hat{A}_0)$, respectively. Here $\Lambda_{0,0}(\cdot)$ is the Dirichlet to Neumann map restricted to $L^2(\partial\Omega)$ (see Section 6.3 for details).

In a second part of the paper we use the definition of perturbation determinants to find trace formulas for extensions, in particular, for pairs of self-adjoint, accumulative and arbitrary closed extensions.

The paper is organized as follows. In Section 2 we give a short introduction into the theory of boundary triplets. Section 3 is devoted to almost solvable extensions. The properties of the perturbation determinant of pairs of proper extensions are investigated and verified in Section 4. Trace formulas are proven in Section 5. In Section 6 we compare the results with those ones of regular pairs. Finally, in Section 7 we give certain examples of perturbation determinants for partial differential operators. In the Appendix we have collected several results for the convenience of the reader which are necessary for proofs or for understanding.

Notation. By \mathfrak{H} and \mathcal{H} we denote separable Hilbert spaces. Linear operators in \mathfrak{H} or \mathcal{H} are always denoted by capital Latin letters, for example by H , A , etc. By $\text{dom}(A)$, $\text{ran}(A)$ and $\rho(A)$ we denote the domain, range and spectrum of A , respectively. The symbols $\sigma_p(\cdot)$, $\sigma_c(\cdot)$ and $\sigma_r(\cdot)$ stand for the point, the continuous and the residual spectrum of a linear operator. Recall that $z \in \rho_c(H)$ if $\ker(H - z) = \{0\}$ and $\text{ran}(H - z) \neq \overline{\text{ran}(H - z)} = \mathfrak{H}$; $z \in \sigma_r(H)$ if $\ker(H - z) = \{0\}$ and $\text{ran}(H - z) \neq \mathfrak{H}$.

The set of bounded linear operators from \mathfrak{H}_1 to \mathfrak{H}_2 is denoted by $[\mathfrak{H}_1, \mathfrak{H}_2]$; $[\mathfrak{H}] := [\mathfrak{H}, \mathfrak{H}]$. The Schatten-v-Neumann ideals of compact operators on a Hilbert space \mathfrak{H} is denoted by $\mathfrak{S}_p(\mathfrak{H})$, $0 < p \leq \infty$; in particular, $\mathfrak{S}_\infty(\mathfrak{H})$ denotes the ideal of all compact operators in \mathfrak{H} . The set of closed linear operators and the set of closed linear relations in \mathcal{H} is denoted by $\mathcal{C}(\mathcal{H})$ and $\tilde{\mathcal{C}}(\mathcal{H})$, respectively.

$C_b^k(\Omega)$, $k \in \mathbb{Z}_+ \cup \{\infty\}$, the set of C^k -functions bounded in Ω with all their derivatives of order $\leq k$, $C_b(\Omega) := C_b^0(\Omega)$; $C_u^k(\Omega)$, $k \in \mathbb{Z}_+ \cup \{\infty\}$, the set of C^k -functions uniformly continuous in Ω with all their derivatives of order $\leq k$, $C_u(\Omega) := C_u^0(\Omega)$; $C_{ub}^k(\Omega) := C_u^k(\Omega) \cap C_b^k(\Omega)$, $C_{ub}(\Omega) := C_{ub}^0(\Omega)$; $H^s(\Omega)$ $s \in \mathbb{R}$, the usual Sobolev spaces.

2 Preliminaries

2.1 Relations

For any linear relation Θ in \mathcal{H} the adjoint relation $\Theta^* \in \tilde{\mathcal{C}}(\mathcal{H})$ is defined by

$$\Theta^* = \left\{ \begin{pmatrix} k \\ k' \end{pmatrix} : (h', k) = (h, k') \text{ for all } \begin{pmatrix} h \\ h' \end{pmatrix} \in \Theta \right\}.$$

A linear relation Θ is called symmetric if $\Theta \subset \Theta^*$ and self-adjoint if $\Theta = \Theta^*$. The relation Θ is called dissipative if $\{h, h'\} \in \Theta$ yields $\text{Im}(h', h) \geq 0$ and accumulative if $-\Theta$ is dissipative. If a dissipative (accumulative) relation Θ does not admit any closed dissipative (accumulative) extension, then Θ is called maximal dissipative or m -dissipative (maximal accumulative or m -accumulative).

We usually consider $\mathcal{C}(\mathcal{H})$ as a subset of $\tilde{\mathcal{C}}(\mathcal{H})$ by identifying an operator $T \in \mathcal{C}(\mathcal{H})$ with its graph $\text{gr}(T)$. In particular, an operator $T \in \mathcal{C}(\mathcal{H})$ is called dissipative if $\text{Im}((Tf, f)) \geq 0$, $f \in \text{dom}(T)$, and accumulative if $\text{Im}(Tf, f) \leq 0$, $f \in \text{dom}(T)$. A dissipative operator T is called maximal dissipative (m -dissipative) if it does not admit any closed dissipative extension.

2.2 Boundary triplets and proper extensions

Let A be a densely defined closed symmetric operator in \mathfrak{H} with equal deficiency indices $n_{\pm}(A) = \dim(\mathfrak{N}_{\mp i})$, $\mathfrak{N}_z := \ker(A^* - z)$.

Definition 2.1.

- (i) A closed extension \tilde{A} of A is called a proper extension, in short $\tilde{A} \in \text{Ext}_A$, if $A \subseteq \tilde{A} \subseteq A^*$;
- (ii) Two proper extensions \tilde{A}' , \tilde{A} are called disjoint if $\text{dom}(\tilde{A}') \cap \text{dom}(\tilde{A}) = \text{dom}(A)$ and transversal if in addition $\text{dom}(\tilde{A}') + \text{dom}(\tilde{A}) = \text{dom}(A^*)$.

Any extension $\tilde{A} = \tilde{A}^*$ of A is proper, $\tilde{A} \in \text{Ext}_A$. Moreover, any dissipative (accumulative) extension \tilde{A} of A is proper, (cf. [34, Theorem III.1.3], [49]). In the following we call also A and A^* be proper extensions. This sounds strange but makes sense with respect of Proposition 2.3 below.

Definition 2.2 ([34]). A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where \mathcal{H} is an auxiliary Hilbert space and $\Gamma_0, \Gamma_1 : \text{dom}(A^*) \rightarrow \mathcal{H}$ are linear mappings, is called an (ordinary) boundary triplet for A^* if the "abstract Green's identity"

$$(A^*f, g) - (f, A^*g) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \quad f, g \in \text{dom}(A^*), \quad (2.1)$$

holds and the mapping $\Gamma := (\Gamma_0, \Gamma_1)^{\top} : \text{dom}(A^*) \rightarrow \mathcal{H} \oplus \mathcal{H}$ is surjective.

A boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* exists whenever $n_+(A) = n_-(A)$. Note also that $n_{\pm}(A) = \dim(\mathcal{H})$ and $\ker(\Gamma_0) \cap \ker(\Gamma_1) = \text{dom}(A)$.

With any boundary triplet Π one associates two canonical self-adjoint extensions $A_j := A^* \upharpoonright \ker(\Gamma_j)$, $j = 0, 1$. Conversely, for any extension $A_0 = A_0^* \in \text{Ext}_A$ there exists a (non-unique) boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* such that $A_0 := A^* \upharpoonright \ker(\Gamma_0)$.

Using the concept of boundary triplets one can parameterize all proper extensions of A in the following way.

Proposition 2.3 ([16, 49]). *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* . Then the mapping*

$$(\text{Ext}_A \ni) \tilde{A} \rightarrow \Gamma \text{dom}(\tilde{A}) = \{ \{ \Gamma_0 f, \Gamma_1 f \} : f \in \text{dom}(\tilde{A}) \} =: \Theta \in \tilde{\mathcal{C}}(\mathcal{H}) \quad (2.2)$$

establishes a bijective correspondence between Ext_A and $\tilde{\mathcal{C}}(\mathcal{H})$. We write $\tilde{A} = A_{\Theta}$ if \tilde{A} corresponds to Θ by (2.2). Moreover, the following holds:

- (i) $A_{\Theta}^* = A_{\Theta^*}$, in particular, $A_{\Theta}^* = A_{\Theta}$ if and only if $\Theta^* = \Theta$.

- (ii) A_Θ is symmetric (self-adjoint) if and only if Θ is symmetric (self-adjoint).
- (iii) A_Θ is m -dissipative (m -accumulative) if and only if so is Θ .
- (iv) The extensions A_Θ and A_0 are disjoint (transversal) if and only if $\Theta \in \mathcal{C}(\mathcal{H})$ ($\Theta \in [\mathcal{H}]$). In this case (2.2) takes the form

$$A_\Theta := A_{\text{gr}}(\Theta) = A^* \upharpoonright \ker(\Gamma_1 - \Theta\Gamma_0). \quad (2.3)$$

In particular, $A_j := A^* \upharpoonright \ker(\Gamma_j) = A_{\Theta_j}$, $j \in \{0, 1\}$, where $\Theta_0 := \{0\} \times \mathcal{H}$ and $\Theta_1 := \mathcal{H} \times \{0\} = \text{gr}(\mathbb{O})$ where \mathbb{O} denotes the zero operator in \mathcal{H} .

We note that $\tilde{\mathcal{C}}(\mathcal{H})$ contains the linear relations $\{0\} \times \{0\}$ and $\mathcal{H} \times \mathcal{H}$. It turns out that the corresponding closed extensions of A are A and A^* , respectively. This is the reason why we include A and A^* into the sets of proper extensions of A .

2.3 Weyl functions and spectra of proper extensions

It is well known that Weyl functions are an important tool in the direct and inverse spectral theory of singular Sturm-Liouville operators. In [15, 16, 19] the concept of Weyl function was generalized to the case of an arbitrary symmetric operator A with $n_+(A) = n_-(A) \leq \infty$. Here following [16] we briefly recall basic facts on Weyl functions and γ -fields associated with a boundary triplet Π .

Definition 2.4 ([15, 16]). Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* and $A_0 = A^* \upharpoonright \ker(\Gamma_0)$. The operator valued functions $\gamma(\cdot) : \rho(A_0) \rightarrow [\mathcal{H}, \mathfrak{H}]$ and $M(\cdot) : \rho(A_0) \rightarrow [\mathcal{H}]$ defined by

$$\gamma(z) := (\Gamma_0 \upharpoonright \mathfrak{N}_z)^{-1} \quad \text{and} \quad M(z) := \Gamma_1 \gamma(z), \quad z \in \rho(A_0), \quad (2.4)$$

are called the γ -field and the Weyl function, respectively, corresponding to the boundary triplet Π .

Clearly, the Weyl function can equivalently be defined by

$$M(z)\Gamma_0 f_z = \Gamma_1 f_z, \quad f_z \in \mathfrak{N}_z, \quad z \in \rho(A_0).$$

The γ -field $\gamma(\cdot)$ and the Weyl function $M(\cdot)$ in (2.4) are well defined. Moreover, both $\gamma(\cdot)$ and $M(\cdot)$ are holomorphic on $\rho(A_0)$ and the following relations hold

$$\gamma(z) = (I + (z - \zeta)(A_0 - z)^{-1})\gamma(\zeta), \quad z, \zeta \in \rho(A_0), \quad (2.5)$$

and

$$M(z) - M(\zeta)^* = (z - \bar{\zeta})\gamma(\zeta)^*\gamma(z), \quad z, \zeta \in \rho(A_0). \quad (2.6)$$

Identity (2.6) yields that $M(\cdot) \in (R_{\mathcal{H}})$, i.e. $M(\cdot)$ is a $[\mathcal{H}]$ -valued Nevanlinna function, that is, $M(\cdot)$ is a $([\mathcal{H}]$ -valued) holomorphic function on $\mathbb{C} \setminus \mathbb{R}$ and

$$M(z) = M(\bar{z})^* \quad \text{and} \quad \frac{\text{Im}(M(z))}{\text{Im}(z)} \geq 0, \quad z \in \rho(A_0).$$

It follows also from (2.6) that $M(\cdot)$ satisfies $0 \in \rho(\text{Im}(M(z)))$ for all $z \in \mathbb{C} \setminus \mathbb{R}$.

Proposition 2.5 ([15, 16]). *Let A be a simple symmetric operator in \mathfrak{H} simple closed densely defined symmetric operator in \mathfrak{H} and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* , $M(\cdot)$ the corresponding Weyl function. Assume that $\Theta \in \tilde{\mathcal{C}}(\mathcal{H})$ and $z \in \rho(A_0)$. Then the following holds:*

- (i) $z \in \rho(A_\Theta)$ if and only if $0 \in \rho(\Theta - M(z))$;
- (ii) $z \in \sigma_\tau(A_\Theta)$ if and only if $0 \in \sigma_\tau(\Theta - M(z))$, $\tau = p, c, r$. Moreover, $\dim(\ker(A_\Theta - z)) = \dim(\ker(\Theta - M(z)))$.

2.4 Krein-type formula for resolvents and comparability

Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* , $M(\cdot)$ and $\gamma(\cdot)$ the corresponding Weyl function and the γ -field, respectively. For any proper (not necessarily self-adjoint) extension $\tilde{A}_\Theta \in \text{Ext}_A$ with non-empty resolvent set $\rho(\tilde{A}_\Theta)$ the following Krein-type formula holds (cf. [15, 16, 19])

$$(A_\Theta - z)^{-1} - (A_0 - z)^{-1} = \gamma(z)(\Theta - M(z))^{-1}\gamma^*(\bar{z}), \quad z \in \rho(A_0) \cap \rho(A_\Theta). \quad (2.7)$$

Formula (2.7) extends the known Krein formula for canonical resolvents to the case of any $A_\Theta \in \text{Ext}_A$ with $\rho(A_\Theta) \neq \emptyset$. Moreover, formulas (2.2), (2.3) and (2.4) express all parameters in (2.7) in terms of the boundary triplet Π (cf. [15, 16, 19]). Namely, these expressions make it possible to apply formula (2.7) to boundary value problems.

The following result is deduced from formula (2.7).

Proposition 2.6 ([16, Theorem 2]). *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* , $\Theta', \Theta \in \tilde{\mathcal{C}}(\mathcal{H})$ and $\rho(A_{\Theta'}) \cap \rho(A_\Theta) \neq \emptyset$. If $\rho(\Theta') \cap \rho(\Theta) \neq \emptyset$, then for any Neumann-Schatten ideal \mathfrak{S}_p , $1 \leq p \leq \infty$, the following holds:*

- (i) *The relation*

$$(A_{\Theta'} - z)^{-1} - (A_\Theta - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}), \quad z \in \rho(A_{\Theta'}) \cap \rho(A_\Theta), \quad (2.8)$$

is equivalent to

$$(\Theta' - \zeta)^{-1} - (\Theta - \zeta)^{-1} \in \mathfrak{S}_p(\mathcal{H}), \quad \zeta \in \rho(\Theta') \cap \rho(\Theta). \quad (2.9)$$

In particular, $(A_\Theta - z)^{-1} - (A_0 - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H})$ for $z \in \rho(A_\Theta) \cap \rho(A_0)$ if and only if $(\Theta - \zeta)^{-1} \in \mathfrak{S}_p(\mathcal{H})$ for $\zeta \in \rho(\Theta)$.

- (ii) *If $\text{dom}(\Theta') = \text{dom}(\Theta)$, then the following implication holds:*

$$\overline{\Theta' - \Theta} \in \mathfrak{S}_p(\mathcal{H}) \implies (A_{\Theta'} - z)^{-1} - (A_\Theta - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}), \quad (2.10)$$

$z \in \rho(A_{\Theta'}) \cap \rho(A_\Theta)$. In particular, if $\Theta', \Theta \in [\mathcal{H}]$, then (2.8) is equivalent to $\Theta' - \Theta \in \mathfrak{S}_p(\mathcal{H})$.

3 Almost solvable extensions

3.1 Basic facts

In the following A denotes a densely defined closed symmetric operator in \mathfrak{H} . The concept of almost solvable extensions of A was introduced in [20] (see also [17, 18, 19]). Let us recall basic facts on these extensions and let us extend this concept to a family of proper extensions.

Definition 3.1.

- (i) An extension $\tilde{A} \in \text{Ext } A$ is called almost solvable if there exists a self-adjoint extension \hat{A} of A such that \hat{A} and \tilde{A} are transversal, see Definition 2.1(ii).
- (ii) The family $\{\tilde{A}_j\}_{j=1}^N$, $\tilde{A}_j \in \text{Ext } A$, $j \in \{1, \dots, N\}$, $2 \leq N \leq \infty$, is called jointly almost solvable if there exists a self-adjoint extension \hat{A} of A such that \hat{A} is transversal to each \tilde{A}_j , $j \in \{1, \dots, N\}$.
- (iii) Let $\lambda \in \mathbb{R}$. The family $\{\tilde{A}_j\}_{j=1}^N$, $\tilde{A}_j \in \text{Ext } A$, $j \in \{1, \dots, N\}$, $2 \leq N \leq \infty$, is called jointly almost solvable with respect to λ if there exists a self-adjoint extension \hat{A} of A which is transversal to any \tilde{A}_j , $j \in \{1, \dots, N\}$, $2 \leq N \leq \infty$, and satisfies in addition the condition $\lambda \in \rho(\hat{A})$.

We note that Definition 3.1(i) coincides with Definition 3 of [18].

Definition 3.2. Let $\tilde{A}_j \in \text{Ext } A$, $j \in \{1, \dots, N\}$. A boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* is called regular for $\{\tilde{A}_j\}_{j=1}^N$ if there exist operators $B_j \in [\mathcal{H}]$, $j \in \{1, \dots, N\}$, such that $\tilde{A}_j = A_{B_j} := A^* \upharpoonright \ker(\Gamma_1 - B_j \Gamma_0)$, $j \in \{1, \dots, N\}$.

Proposition 3.3. Let $\tilde{A}_j \in \text{Ext } A$, $j \in \{1, \dots, N\}$.

- (i) The family $\{\tilde{A}_j\}_{j=1}^N$ is jointly almost solvable if and only if there exists a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* which is regular for $\{\tilde{A}_j\}_{j=1}^N$.
- (ii) The family $\{\tilde{A}_j\}_{j=1}^N$ is jointly almost solvable with respect to $\lambda \in \mathbb{R}$ if and only if there exists a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* which is regular for $\{\tilde{A}_j\}_{j=1}^N$ and $\lambda \in \rho(A_0)$.

Proof. (i) The proof follows immediately from Proposition 2.3(iv) and [19, Proposition 7.1].

(ii) By Definition 3.1(iii) there exists a self-adjoint extension \hat{A} which is transversal to \tilde{A}_j , $j \in \{1, 2, \dots, N\}$, and such that $\lambda \in \rho(\hat{A})$. Choosing a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* such that $\hat{A} := A_0 = A^* \upharpoonright \ker(\Gamma_0)$ and applying Proposition 2.3(iv) we get the necessity. Sufficiency is again implied by [19, Proposition 7.1]. \square

Proposition 3.3(i) makes it possible to introduce the real part and the imaginary part of an almost solvable extension \tilde{A} .

Definition 3.4. [20] Let $\tilde{A} \in \text{Ext}_A$ be almost solvable and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* which is regular for \tilde{A} , i.e. $\tilde{A} = A_B$, $B \in [\mathcal{H}]$. Then the self-adjoint extensions $\tilde{A}_R, \tilde{A}_I \in \text{Ext}_A$ defined by $\tilde{A}_R := \tilde{A}_{B_R}$ and $\tilde{A}_I := \tilde{A}_{B_I}$, are called the real and the imaginary parts of \tilde{A} , respectively.

It can be shown (see [17], [19]) that the definitions of \tilde{A}_R and \tilde{A}_I depend only on \tilde{A} and do not depend on the choice of the regular boundary triplet.

It follows from Proposition 3.3 that in the case $n_+(A) = n_-(A) < \infty$ any $\tilde{A} \in \text{Ext}_A$ is almost solvable. The following statement demonstrates that for $n_{\pm}(A) = \infty$ the class of almost solvable extensions is also rather wide.

Proposition 3.5 ([18, Theorem 1]). *Let $\tilde{A} \in \text{Ext}_A$. If the condition $(\rho(\tilde{A}) \cup \sigma_c(\tilde{A})) \cap \mathbb{C}_{\pm} \neq \emptyset$ is satisfied, then \tilde{A} is almost solvable. In particular, \tilde{A} is almost solvable if $\rho(\tilde{A}) \cap \mathbb{R} \neq \emptyset$.*

Recall that an extension $\tilde{A} \in \text{Ext}_A$ is called solvable if $0 \in \rho(\tilde{A})$. Hence any solvable extension is almost solvable. Furthermore, we note that the sufficient condition of Proposition 3.5 is not necessary. It might even happen that $n_+(A_+) = n_-(A_-) < \infty$ and \tilde{A} is almost solvable although $(\rho(\tilde{A}) \cup \sigma_c(\tilde{A})) \cap \mathbb{C}_{\pm} = \emptyset$. Such extensions can easily be constructed for $A = A_+ \oplus A_-$ where A_{\pm} are simple symmetric operators with deficiency indices $n_+(A_+) = n_-(A_-) = 1$ and $n_-(A_+) = n_+(A_-) = 0$.

Finally, we indicate a criteria which easily follows from [17, Proposition 1.5].

Lemma 3.6. *Let A be as above and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* , $M(\cdot)$ the corresponding Weyl function. Let $\tilde{A}' := A_{\Theta'}$ and $\tilde{A} := A_{\Theta}$ where $\Theta', \Theta \in \tilde{\mathcal{C}}(\mathcal{H})$ and $\zeta \in \rho(\tilde{A}') \cap \rho(\tilde{A})$. Then the extensions \tilde{A}' and \tilde{A} are transversal if and only if*

$$0 \in \rho((\Theta' - M(\zeta))^{-1} - (\Theta - M(\zeta))^{-1}).$$

3.2 Compactness and almost solvable extensions

In general, even two almost solvable extensions are not necessarily jointly almost solvable. However, the following result, which is very important in applications to the perturbation determinants, is valid.

Proposition 3.7. *Let A be a densely defined closed symmetric operator and let $\tilde{A}_j \in \text{Ext}_A$, $j \in \{1, \dots, N\}$, $2 \leq N \leq \infty$. If at least one \tilde{A}_{j_0} , $j_0 \in \{1, 2, \dots, N\}$, is almost solvable and there exists a non-real $\zeta \in \bigcap_{j=1}^N \rho(\tilde{A}_j)$ such that*

$$(\tilde{A}_j - \zeta)^{-1} - (\tilde{A}_{j_0} - \zeta)^{-1} \in \mathfrak{S}_{\infty}(\mathfrak{H}), \quad j \in \{1, 2, \dots, N\}, \quad (3.1)$$

then the family $\{\tilde{A}_j\}_{j=1}^N$ is jointly almost solvable.

Proof. Without loss of generality we assume $j_0 = 1$. Since \tilde{A}_1 is almost solvable there is a self-adjoint extension \hat{A} of A which is transversal to \tilde{A}_1 . We choose a

boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* such that $\widehat{A} = A_0 := A^* \upharpoonright \ker(\Gamma_0)$ and denote by $M(\cdot)$ the Weyl function. By Proposition 3.3, there exists a $B_1 \in [\mathcal{H}]$ such that $\widetilde{A}_1 = A_{B_1}$. Since $\zeta \in \rho(\widetilde{A}_1)$ one gets from Proposition 2.5 that $0 \in \rho(B_1 - M(\zeta))$ which yields the existence of the operator $D_1 := (B_1 - M(\zeta))^{-1}$. Moreover, for any $j = 2, 3, \dots, N$, there are closed relations Θ_j in \mathcal{H} such that $\widetilde{A}_j = A_{\Theta_j}$, cf. Proposition 3.3. Again, by Proposition 2.5, $0 \in \rho(\Theta_j - M(\zeta))$ which shows that $D_j := (\Theta_j - M(\zeta))^{-1}$, $j \in \{2, 3, \dots, N\}$, exists and is bounded. From condition (3.1) we get that

$$D_j - D_1 \in \mathfrak{S}_\infty(\mathcal{H}), \quad j = 2, 3, \dots, N. \quad (3.2)$$

Notice, that D_1 is invertible, that is $0 \in \rho(D_1)$.

Without loss of generality we assume $\zeta \in \mathbb{C}_+$. We set

$$B_\mu := M_R(\zeta) + \mu^{-1}M_I(\zeta), \quad \mu \in \mathbb{R} \setminus \{0\},$$

where $M_R(\zeta) := \operatorname{Re}(M(\zeta))$ and $M_I(\zeta) := \operatorname{Im}(M(\zeta))$. The operators B_μ are bounded and self-adjoint. Hence $\widehat{A}_\mu := A_{B_\mu}$ defines a family of self-adjoint extensions of A . Obviously, we have

$$(B_\mu - M(\zeta))^{-1} = \frac{\mu}{1 - i\mu} \frac{1}{M_I(\zeta)}$$

where we have used that for non-real $z \in \mathbb{C}_+$ the operator $M_I(z)$ is invertible. Since $M_I(z)$ is also non-negative we have

$$D_j - (B_\mu - M(\zeta))^{-1} = \frac{1}{\sqrt{M_I(\zeta)}} \left(D'_j - \frac{\mu}{1 - i\mu} \right) \frac{1}{\sqrt{M_I(\zeta)}}, \quad j = 1, 2, \dots, N,$$

where $D'_j := \sqrt{M_I(\zeta)} D_j \sqrt{M_I(\zeta)}$, $j = 1, 2, \dots, N$. From (3.2) we immediately get that

$$D'_j - D'_1 \in \mathfrak{S}_\infty(\mathcal{H}), \quad j = 2, 3, \dots, N.$$

Since D_1 is invertible the operator D'_1 is also invertible, that is $0 \in \rho(D'_1)$. Hence there is a neighborhood \mathcal{U} of zero such that $\mathcal{U} \subseteq \rho(D'_1)$. The set $\Sigma_j := \sigma(D'_j) \cap \mathcal{U}$ consists of isolated eigenvalues of D'_j of finite algebraic multiplicity, that is, the set Σ_j is countable for each $j = 2, 3, \dots, N$. Hence the set $\Sigma := \bigcup_{j=2}^N \Sigma_j$ is countable. Setting $\zeta(\mu) := \mu(1 - i\mu)^{-1}$, $\mu \in \mathbb{R} \setminus \{0\}$, one has $\lim_{\mu \rightarrow 0} \zeta(\mu) = 0$. Since the curve $\zeta(\mu)$ is continuous there is a least one $\mu_0 \in \mathbb{R} \setminus \{0\}$ such that $\zeta(\mu_0) \in \mathcal{U} \setminus \Sigma \subseteq \mathcal{U} \setminus \Sigma_j$, $j = 2, 3, \dots, N$, which yields $\zeta(\mu_0) \in \mathcal{U} \cap \rho(D'_j)$, $j = 2, 3, \dots, N$, and, of course, $\zeta(\mu_0) \in \mathcal{U}$. Hence the operators $D'_j - \zeta(\mu_0)$, $j = 1, 2, \dots, N$, are invertible which shows that $0 \in \rho(D_j - (B_{\mu_0} - M(\zeta))^{-1})$, $j = 1, 2, \dots, N$. Hence $0 \in \rho((\Theta_j - M(\zeta))^{-1} - (B_{\mu_0} - M(\zeta))^{-1})$, $j = 1, 2, \dots, N$. Taking into account Lemma 3.6 we complete the proof. \square

Proposition 3.8. *Let A be a densely defined closed symmetric operator and let $\widetilde{A}_j \in \operatorname{Ext}_A$, $1 \leq j \leq N < \infty$. If there is a real λ such that $\lambda \in \rho(\widetilde{A}_{j_0})$ for some $j_0 \in \{1, 2, \dots, N\}$ and $\zeta \in \bigcap_{j=1}^N \rho(\widetilde{A}_j)$ such that (3.1) holds, then the family $\{\widetilde{A}_j\}_{j=1}^N$ is jointly almost solvable with respect to λ .*

Proof. Without loss of generality we assume $j_0 = 1$. Since $\lambda \in \rho(\tilde{A}_1)$ one gets that λ is a regular point of A . Hence there is a neighborhood $\delta := (\lambda - \epsilon, \lambda + \epsilon)$, $\epsilon > 0$, of λ such that $\delta \subseteq \rho(\tilde{A}_1)$ is a gap for A , that is,

$$\|(A - \lambda)f\| \geq \epsilon \|f\|, \quad f \in \text{dom}(A).$$

From [39] we get that there is a self-adjoint extension \hat{A} of A such that $\rho(\hat{A}) \supseteq \delta$. We choose a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ of A^* with Weyl function $M(z)$ such that $\hat{A} = A_0 = A^* \upharpoonright \ker(\Gamma_0)$. We recall that the Weyl function $M(z)$ is well-defined and bounded on δ because δ is a gap of $\hat{A} = A_0$. Let $B_\mu = M(\lambda) + \mu^{-1}$ where $\mu \in \mathbb{R} \setminus \{0\}$. The operator B_μ is bounded and self-adjoint for $\mu \in \mathbb{R} \setminus \{0\}$. We set $\hat{A}_\mu := A_{B_\mu}$. Obviously, \hat{A}_μ is self-adjoint. Moreover, we have $B_\mu - M(\lambda) = \mu^{-1}$. Hence $(B_\mu - M(\lambda))^{-1} = \mu$ which yields that $0 \in \rho(B_\mu - M(\lambda))$ for any $\mu \in \mathbb{R} \setminus \{0\}$. Using Proposition 2.5 we find $\lambda \in \rho(\hat{A}_\mu)$ for any $\mu \neq 0$.

By Proposition 2.3 we find a closed linear relation Θ_1 in \mathcal{H} such that $\tilde{A}_1 = A_{\Theta_1}$. Moreover, from Proposition 2.5 we get that $0 \in \rho(\Theta_1 - M(\lambda))$. Hence $D_1 := (\Theta_1 - M(\lambda))^{-1}$ is a bounded operator. Furthermore, we have

$$(\Theta_1 - M(\lambda))^{-1} - (B_\mu - M(\lambda))^{-1} = D_1 - \mu.$$

Obviously, there is a $\mu \in \mathbb{R} \setminus \{0\}$ such that $\mu \in \rho(D_1)$ holds. Hence $0 \in \rho((\Theta_1 - M(\lambda))^{-1} - (B_\mu - M(\lambda))^{-1})$. Applying Lemma 3.6 we obtain that the extensions \hat{A}_1 and \hat{A} are transversal for sufficiently large μ .

By assumption (3.1) the sets $\Sigma_j := \sigma(\tilde{A}_j) \cap \delta$, $j = 2, 3, \dots, N$, are countable which yields that $\Sigma = \bigcup_{j=2}^N \Sigma_j$ is countable. Moreover, by Proposition 2.3 there are closed relations Θ_j in \mathcal{H} such that $\tilde{A}_j = A_{\Theta_j}$, $j = 2, 3, \dots, N$. Let $\lambda' \in (\lambda, \lambda + \epsilon) \setminus \Sigma \subseteq (\lambda, \lambda + \epsilon) \setminus \Sigma_j$, $j = 2, 3, \dots, N$, which yields $\lambda' \in (\lambda, \lambda + \epsilon) \cap \rho(\tilde{A}_j)$, $j = 2, 3, \dots, N$, and $\lambda' \in \rho(\tilde{A}_1)$. From Proposition 2.5 we find $0 \in \rho(\Theta_j - M(\lambda'))$ for $j = 1, 2, \dots, N$. Hence the operators $D'_j := (\Theta_j - M(\lambda'))^{-1}$, $j = 2, 3, \dots, N$, exist and are bounded. We set $B'_\mu := M(\lambda') + \mu^{-1}$ and $\hat{A}'_\mu := A_{B'_\mu}$, $\mu \in \mathbb{R} \setminus \{0\}$. Obviously, \hat{A}'_μ is self-adjoint. We have

$$(\Theta_j - M(\lambda'))^{-1} - (B'_\mu - M(\lambda'))^{-1} = D'_j - \mu, \quad j = 1, 2, \dots, N.$$

Hence, there is a sufficiently large real number $\mu_0 > 0$ such that $0 \in \rho((\Theta_j - M(\lambda'))^{-1} - (B'_{\mu_0} - M(\lambda'))^{-1})$, $j = 1, 2, 3, \dots, N$. By Lemma 3.6 the self-adjoint extension \hat{A}'_{μ_0} is transversal to \tilde{A}_j , $j = 1, 2, 3, \dots, N$.

It remains to show that $\lambda \in \rho(\hat{A}'_{\mu_0})$. We have $B'_{\mu_0} - M(\lambda) = \mu_0^{-1} + M(\lambda') - M(\lambda) \geq \mu_0^{-1}$ where we have used that $M(\lambda) \leq M(\lambda')$ for $\lambda, \lambda' \in \delta$ and $\lambda < \lambda'$. Hence $0 \in \rho(B'_{\mu_0} - M(\lambda))$. From Proposition 2.5 we obtain $\lambda \in \hat{A}'_{\mu_0}$. \square

3.3 Characteristic function and almost solvable extensions

It is known several approaches to the definition of the characteristic function (CF) of an unbounded operator with non-empty resolvent set. The most relevant

to our considerations definitions have been proposed in [60] and [17, 18]. In general, the CF might have some exotic properties. However, it was shown in [17, 18, 20] that the CF of an almost solvable extension of A takes values in $[\mathcal{H}]$ and has some nice properties similar to that of the CF of bounded operators (cf. [13]). We will not present a strict definition of CF since in what follows we need only the following formula expressed CF in terms of the Weyl function.

Proposition 3.9 ([17, Theorem 2]). *Let A be a densely defined closed symmetric operator and let \tilde{A} be an almost solvable extension of A . Let also $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* which is regular for \tilde{A} , i.e. $\tilde{A} = A_B = A^* \upharpoonright \ker(\Gamma_1 - B\Gamma_0)$ and $B \in [\mathcal{H}]$. Then the characteristic function of the operator A_B admits the representation*

$$W_A^\Pi(z) := I + 2i|B_I|^{1/2}(B^* - M(z))^{-1}|B_I|^{1/2}J, \quad z \in \rho(\tilde{A}^*) \cap \rho(A_0), \quad (3.3)$$

where $B_I = J|B_I|$, $J = \text{sign}(B_I)$, is the polar decomposition of $B_I := \text{Im}(B)$.

It follows from (3.3) that $W_A^\Pi(\cdot)$ takes values in $[\mathcal{H}]$ and is J -contractive in \mathbb{C}_+ , respectively J -expansive in \mathbb{C}_- . In particular, it is contractive in \mathbb{C}_+ if $\tilde{A} = A_B$ is m -dissipative, that is, B is m -dissipative, cf. Proposition 2.3(iii). Notice that $J = I$ in this case.

4 Perturbation determinants for extensions

4.1 Elementary properties

Through this section we always assume that A is a densely defined closed symmetric operator in \mathfrak{H} with equal deficiency indices. We are going to show that the perturbation determinant for extensions $\Delta_{\tilde{A}'/\tilde{A}}^\Pi(\cdot)$, cf. Definition 1.1, has similar properties as the perturbation determinant for additive perturbations $\Delta_{H'/H}(\cdot)$.

Lemma 4.1. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* , $M(\cdot)$ the corresponding Weyl function. Let also $\tilde{A}', \tilde{A} \in \text{Ext}_A$ and $\tilde{A}' = A_{B'}$ and $\tilde{A} = A_B$. If $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$, then the following holds:*

- (i) $\{z \in \rho(A_0) : 0 \in \rho(B - M(z))\} = \rho(\tilde{A}) \cap \rho(A_0) \neq \emptyset$ and (1.6) holds for $\zeta \in \rho(\tilde{A}') \cap \rho(\tilde{A})$.
- (ii) The perturbation determinant $\Delta_{\tilde{A}'/\tilde{A}}^\Pi(z)$ is well defined on the open set $\rho(\tilde{A}) \cap \rho(A_0)$ and is holomorphic there.
- (iii) If Π is regular for $\{\tilde{A}', \tilde{A}\}$, then $B', B \in [\mathcal{H}]$ and $B' - B \in \mathfrak{S}_1(\mathcal{H})$.
- (iv) If $n_\pm(A) < \infty$, then

$$\Delta_{\tilde{A}'/\tilde{A}}^\Pi(z) = \frac{\det(B' - M(z))}{\det(B - M(z))}, \quad z \in \rho(\tilde{A}) \cap \rho(A_0). \quad (4.1)$$

Proof. (i) This statement follows immediately from Proposition 2.5(i).

To prove (1.6) we start with the identity

$$(B - M(z))^{-1} - (B' - M(z))^{-1} = (B' - M(z))^{-1}(B' - B)(B - M(z))^{-1} \quad (4.2)$$

valid for $z \in \rho(\tilde{A}') \cap \rho(\tilde{A})$. It follows that the right-hand side is a trace class operator due to the assumption $(B' - B)(B - M(z))^{-1} \in \mathfrak{S}_1(\mathcal{H})$. Now (1.6) is implied by combining Krein-type formula (2.7) with the identity (4.2).

(ii) By Proposition 2.5 we have $0 \in \rho(B - M(z))$ if $z \in \rho(A_B) \cap \rho(A_0)$. Taking into account Definition 1.1 we find that the perturbation determinant is defined on $\rho(\tilde{A}) \cap \rho(A_0)$. Moreover, since the operator-valued function $(B - M(z))^{-1}$ is holomorphic on the open set $\{z \in \mathbb{C} : 0 \in \rho(B - M(z))\}$ the perturbation determinant $\Delta_{\tilde{A}'/\tilde{A}}^\Pi(\cdot)$ is holomorphic on $z \in \rho(A_B) \cap \rho(A_0)$.

(iii) If Π is regular for $\{\tilde{A}', \tilde{A}\}$, then $B', B \in [\mathcal{H}]$ by definition. Therefore, by Proposition 2.6(ii), the inclusion (1.6) is equivalent to $B' - B \in \mathfrak{S}_1(\mathcal{H})$.

(iv) If $n_\pm(A) < \infty$, then $\dim(\mathcal{H}) = n_\pm(A) < \infty$. Hence B' and B are bounded operators and (4.1) is implied by combining the identity

$$I_{\mathcal{H}} + (B' - B)(B - M(z))^{-1} = (B' - M(z))(B - M(z))^{-1}, \quad z \in \rho(\tilde{A}') \cap \rho(A_0).$$

with Proposition A.2(ii). \square

Remark 4.2. Lemma 4.1 rises several problems.

(a) By Lemma 4.1(i) the assumption $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$ yields (1.6). Is the converse true? In other words, is there exists a boundary triplet Π for A^* such that $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$ whenever the extensions $\tilde{A}', \tilde{A} \in \text{Ext}_A$ satisfy condition (1.6)? The answer is not obvious since due to (4.2) the inclusion (1.6) is in general implied by the inclusion $(B' - B)(B - M(z))^{-1} \in \mathfrak{S}_1(\mathcal{H})$ but not vice versa.

(b) The perturbation determinant $\Delta_{\tilde{A}'/\tilde{A}}^\Pi(\cdot)$ depends on a chosen boundary triplet Π . What is character of this dependence?

(c) In vie of Lemma 4.1(ii), $\Delta_{\tilde{A}'/\tilde{A}}^\Pi(\cdot)$ is holomorphic on $\rho(\tilde{A}) \cap \rho(A_0)$. We will show the next section that $\Delta_{\tilde{A}'/\tilde{A}}^\Pi(\cdot)$ admits holomorphic continuation to the domain $\rho(\tilde{A})$, as it takes place in the classical definition.

4.2 Existence of an appropriate boundary triplet

We are going to answer problem (a) of Remark 4.2, at least, partially.

Proposition 4.3. *Let $\tilde{A}', \tilde{A} \in \text{Ext}_A$, $\rho(\tilde{A}') \cap \rho(\tilde{A}) \neq \emptyset$ and let (1.6) be valid. If \tilde{A} is an almost solvable extension, then there exists a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* such that $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$. Moreover, the boundary triplet Π can be chosen regular for $\{\tilde{A}', \tilde{A}\}$.*

Proof. By Proposition 3.7, the pair $\{\tilde{A}', \tilde{A}\}$ is jointly almost solvable. By Proposition 3.3(i), we can find a regular boundary triplet Π . Finally, using Lemma 4.1(iii) we get $B' - B \in \mathfrak{S}_1(\mathcal{H})$ which yields $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$. \square

Corollary 4.4. *Let $\tilde{A}', \tilde{A} \in \text{Ext}_A$, $\rho(\tilde{A}') \cap \rho(\tilde{A}) \neq \emptyset$ and let (1.6) be valid.*

(i) *If $(\rho(\tilde{A}) \cup \sigma_c(\tilde{A})) \cap \mathbb{C}_+$ and $(\rho(\tilde{A}) \cup \sigma_c(\tilde{A})) \cap \mathbb{C}_-$ are not empty, then there exists a boundary triplet Π for A^* which is regular for $\{\tilde{A}', \tilde{A}\}$ and such that $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$.*

(ii) *If $\rho(\tilde{A}) \cap \mathbb{R}$ is not empty, then there exists a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* such that $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$. Moreover, this triplet can be chosen regular for $\{\tilde{A}', \tilde{A}\}$ such that the condition $\lambda \in \rho(A_0)$, $A_0 := A^* \upharpoonright \ker(\Gamma_0)$, for some $\lambda \in \rho(\tilde{A}) \cap \mathbb{R}$ is satisfied.*

Proof. (i) The proof follows from Proposition 3.5, Proposition 3.7, Proposition 3.3(i) and Proposition 4.3.

(ii) From Proposition 3.5 and Proposition 3.8 we get that $\{\tilde{A}', \tilde{A}\}$ is almost solvable with respect to λ . Proposition 3.3(ii) and Proposition 4.3 yield the existence of the desired boundary triplet. \square

4.3 Dependence on the boundary triplet

Let us answer problem (b) of Remark 4.2.

Proposition 4.5. *Let $\tilde{A}', \tilde{A} \in \text{Ext}_A$ and let $\Pi = \{\mathcal{H}, \Gamma_1, \Gamma_0\}$ and $\Pi' = \{\mathcal{H}', \Gamma'_1, \Gamma'_0\}$ be boundary triplets for A^* such that $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$ and $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^{\Pi'}$. If \tilde{A} is almost solvable, then there exists a constant $c \in \mathbb{C}$ such that*

$$\Delta_{\tilde{A}'/\tilde{A}}^\Pi(z) = c \Delta_{\tilde{A}'/\tilde{A}}^{\Pi'}(z), \quad z \in \rho(\tilde{A}) \cap \rho(A_0), \quad (4.3)$$

where $A_0 = A^* \upharpoonright \ker(\Gamma_0)$ and $A'_0 = A^* \upharpoonright \ker(\Gamma'_0)$. If \tilde{A}' and \tilde{A} are self-adjoint, then c is real.

Proof. By Proposition 4.3 there exists a boundary triplet $\tilde{\Pi} = \{\tilde{\mathcal{H}}, \tilde{\Gamma}_0, \tilde{\Gamma}_1\}$ for A^* such that $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^{\tilde{\Pi}}$ and which is regular for $\{\tilde{A}', \tilde{A}\}$. Since $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$ and $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^{\tilde{\Pi}}$ there exist operators $B', B \in \mathcal{C}(\mathcal{H})$ and $\tilde{B}', \tilde{B} \in [\tilde{\mathcal{H}}]$ which satisfy the requirements of Definition 1.1. In particular, one has $\tilde{A}' = A_{B'} = A_{\tilde{B}'}$ and $\tilde{A} := A_B = A_{\tilde{B}}$. Since $\tilde{\Pi}$ is regular we have $\tilde{B}' - \tilde{B} \in \mathfrak{S}_1(\tilde{\mathcal{H}})$ (cf. Lemma 4.1). Moreover, we have

$$\begin{aligned} \Delta_{\tilde{A}'/\tilde{A}}^\Pi(z) &= \det(I + (B' - B)(B - M(z))^{-1}), \quad z \in \rho(\tilde{A}) \cap \rho(A_0), \\ \Delta_{\tilde{A}'/\tilde{A}}^{\tilde{\Pi}}(z) &= \det(I + (\tilde{B}' - \tilde{B})(\tilde{B} - \tilde{M}(z))^{-1}), \quad z \in \rho(\tilde{A}) \cap \rho(\tilde{A}_0), \end{aligned}$$

where $\tilde{A}_0 = A^* \upharpoonright \ker(\tilde{\Gamma}_0)$.

By [34] (see also [19, Proposition 1.7]) there exists a J -unitary block-operator matrix $X \in [\mathcal{H} \oplus \mathcal{H}]$ such that

$$\begin{pmatrix} \tilde{\Gamma}_1 \\ \tilde{\Gamma}_0 \end{pmatrix} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ \Gamma_0 \end{pmatrix}$$

where $J := \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}$. The corresponding Weyl functions as well as boundary operators are related by (see [19, Proposition 1.7])

$$\begin{aligned}\tilde{B} &= (X_{11}B + X_{12})(X_{21}B + X_{22})^{-1}, \\ \tilde{B}' &= (X_{11}B' + X_{12})(X_{21}B' + X_{22})^{-1}.\end{aligned}\tag{4.4}$$

Since \tilde{B}' and \tilde{B} are bounded operators it follows from [19, Proposition 1.7] that $0 \in \rho(X_{21}B + X_{22})$ and $0 \in \rho(X_{21}B' + X_{22})$. In particular, one has that $\text{dom}(X_{21}B + X_{22}) = \text{dom}(B)$ and $\text{dom}(B) = \text{ran}((X_{21}B + X_{22})^{-1})$. Similarly one gets that $0 \in \rho(X_{21}M(z) + X_{22})$ and

$$\widetilde{M}(z) = (X_{11}M(z) + X_{12})(X_{21}M(z) + X_{22})^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}.\tag{4.5}$$

Taking this fact into account we have

$$\begin{aligned}\tilde{B} - \widetilde{M}(z) &= (X_{11}B + X_{12})(X_{21}B + X_{22})^{-1} - (X_{11}M(z) + X_{12})(X_{21}M(z) + X_{22})^{-1} \\ &= X_{11}(B - M(z))(X_{21}B + X_{22})^{-1} + \\ &\quad (X_{11}M(z) + X_{12}) \left\{ (X_{21}B + X_{22})^{-1} - (X_{21}M(z) + X_{22})^{-1} \right\} \\ &= X_{11}(B - M(z))(X_{21}B + X_{22})^{-1} - \\ &\quad (X_{11}M(z) + X_{12})(X_{21}M(z) + X_{22})^{-1}X_{21}(B - M(z))(X_{21}B + X_{22})^{-1}.\end{aligned}$$

Hence

$$\tilde{B} - \widetilde{M}(z) = Q(z)(B - M(z))(X_{21}B + X_{22})^{-1}, \quad z \in \mathbb{C}_{\pm},$$

where

$$Q(z) := X_{11} - (X_{11}M(z) + X_{12})(X_{21}M(z) + X_{22})^{-1}X_{21}$$

which is a well defined operator-valued function. This yields the representation

$$Q(z) = (\tilde{B} - \widetilde{M}(z))(X_{21}B + X_{22})(B - M(z))^{-1}, \quad z \in \rho(\tilde{A}) \cap \mathbb{C}_{\pm}.$$

We set

$$\Xi(z) := (B - M(z))(X_{21}B + X_{22})^{-1}(\tilde{B} - \widetilde{M}(z))^{-1}, \quad z \in \rho(\tilde{A}) \cap \rho(\tilde{A}_0),$$

and note that due to (4.4) $\Xi(\cdot)$ is a well-defined family of bounded operators. Clearly, $Q(z)\Xi(z) = I_{\tilde{\mathcal{H}}}$ and $\Xi(z)Q(z) = I_{\mathcal{H}}$ for $z \in \rho(\tilde{A}) \cap \mathbb{C}_{\pm}$. Hence

$$(\tilde{B} - \widetilde{M}(z))^{-1} = (X_{21}B + X_{22})(B - M(z))^{-1}\Xi(z), \quad z \in \rho(\tilde{A}) \cap \mathbb{C}_{\pm}.\tag{4.6}$$

Similarly we find

$$\tilde{B}' - \widetilde{M}'(z) = Q(z)(B' - M(z))(X_{21}B' + X_{22})^{-1}, \quad z \in \mathbb{C}_{\pm}.\tag{4.7}$$

Combining (4.6) with (4.7) we arrive at the representation

$$\begin{aligned} I + (\tilde{B}' - \tilde{B})(\tilde{B} - \tilde{M}(z))^{-1} \\ = Q(z)(B' - M(z))(X_{21}B' + X_{22})^{-1}(X_{21}B + X_{22})(B - M(z))^{-1}\Xi(z) \end{aligned}$$

for $z \in \rho(\tilde{A}) \cap \mathbb{C}_\pm$. It follows that

$$\begin{aligned} I + (\tilde{B}' - \tilde{B})(\tilde{B} - \tilde{M}(z))^{-1} \\ = Q(z)(B' - M(z)) \times \\ \left\{ I + (X_{21}B' + X_{22})^{-1}X_{21}(B - B') \right\} (B - M(z))^{-1}\Xi(z) \\ = Q(z)(B' - M(z))(B - M(z))^{-1}\Xi(z) + \\ Q(z)(B' - M(z))(X_{21}B' + X_{22})^{-1}X_{21}(B - B')(B - M(z))^{-1}\Xi(z) \\ = I + Q(z)(B' - B)(B - M(z))^{-1}Q(z)^{-1} + \\ Q(z)(B' - M(z))(X_{21}B' + X_{22})^{-1}X_{21}(B - B')(B - M(z))^{-1}Q(z)^{-1}. \end{aligned}$$

Hence

$$\begin{aligned} \det(I + (\tilde{B}' - \tilde{B})(\tilde{B} - \tilde{M}(z))^{-1}) \\ = \det \left(I + (B' - B)(B - M(z))^{-1} \right. \\ \left. + (B' - M(z))(X_{21}B' + X_{22})^{-1}X_{21}(B - B')(B - M(z))^{-1} \right). \end{aligned}$$

Further, it is easily seen that

$$\begin{aligned} I + (B' - B)(B - M(z))^{-1} + \\ (B' - M(z))(X_{21}B' + X_{22})^{-1}X_{21}(B - B')(B - M(z))^{-1} \\ = \left(I + (B' - B)(B - M(z))^{-1} \right) \times \\ \times \left(I + (B - M(z))(X_{21}B' + X_{22})^{-1}X_{21}(B - B')(B - M(z))^{-1} \right). \end{aligned}$$

By the multiplicative property for determinants we get

$$\begin{aligned} \det(I + (\tilde{B}' - \tilde{B})(\tilde{B} - \tilde{M}(z))^{-1}) \\ = \det(I + (B' - B)(B - M(z))^{-1}) \det(I + X_{21}(B - B')(X_{21}B' + X_{22})^{-1}). \end{aligned}$$

Note, that the second determinant in the last formula is well defined. Indeed, it follows from (4.7) that the operator valued function

$$(B' - M(z))(X_{21}B' + X_{22})^{-1} = Q(z)^{-1}(\tilde{B}' - M(z))$$

takes values in \mathcal{H} for $z \in \rho(\tilde{A}') \cap \mathbb{C}_\pm$. Taking into account Definition 1.1(iii) we get $(B - B')(B' - M(z))^{-1} \in \mathfrak{S}_1(\mathcal{H})$, $z \in \rho(\tilde{A}') \cap \mathbb{C}_\pm$. Therefore

$$\begin{aligned} X_{21}(B - B')(X_{21}B' + X_{22})^{-1} \\ = X_{21}(B - B')(B' - M(z))^{-1}(B' - M(z))(X_{21}B' + X_{22})^{-1} \in \mathfrak{S}_1(\mathcal{H}) \end{aligned}$$

for $z \in \rho(\tilde{A}') \cap \mathbb{C}_\pm$ and the determinant is well-defined. Setting $\tilde{c} := \det \left(I + X_{21}(B - B')(X_{21}B' + X_{22})^{-1} \right)$ we get $\Delta_{\tilde{A}'/\tilde{A}}^{\tilde{\Pi}}(z) = \tilde{c} \Delta_{\tilde{A}'/\tilde{A}}^{\Pi}(z)$ for $z \in \rho(\tilde{A}) \cap \mathbb{C}_\pm$.

Similarly, there exists a constant $\tilde{c}' \in \mathbb{C}$ such that $\Delta_{\tilde{A}'/\tilde{A}}^{\tilde{\Pi}}(z) = \tilde{c}' \Delta_{\tilde{A}'/\tilde{A}}^{\Pi'}(z)$ for $z \in \rho(\tilde{A}) \cap \mathbb{C}_\pm$. Setting $c := \tilde{c}'/\tilde{c}$ we find $\Delta_{\tilde{A}'/\tilde{A}}^{\Pi}(z) = c \Delta_{\tilde{A}'/\tilde{A}}^{\Pi'}(z)$ for $z \in \rho(\tilde{A}) \cap \mathbb{C}_\pm$.

It remains to prove that $c = \bar{c}$ whenever $\tilde{A}' = (\tilde{A}')^*$ and $\tilde{A} = (\tilde{A})^*$. We have

$$c \Delta_{\tilde{A}'/\tilde{A}}^{\Pi'}(\bar{z}) = \Delta_{\tilde{A}'/\tilde{A}}^{\Pi}(\bar{z}) = \overline{\Delta_{\tilde{A}'/\tilde{A}}^{\Pi}(z)} = \bar{c} \Delta_{\tilde{A}'/\tilde{A}}^{\Pi'}(\bar{z}), \quad z \in \mathbb{C}_\pm,$$

which yields $c = \bar{c}$.

Finally we note that (4.3) was proved for $z \in \rho(\tilde{A}) \cap \mathbb{C}_\pm$. Since both sides admit an analytic continuation to $\rho(\tilde{A}) \cap \rho(A_0) \cap \rho(A_0')$, the relation (4.3) extends to this set too. \square

4.4 Domain of holomorphy

We note that relation (4.3) is not completely satisfactory because it should be valid for $z \in \rho(\tilde{A})$. To this end we have to answer problem (c) of Remark 4.2.

Proposition 4.6. *Let $\tilde{A}', \tilde{A} \in \text{Ext}_A$ and let Π be a boundary triplet for A^* such that $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$. Then $\Delta_{\tilde{A}'/\tilde{A}}^{\Pi}(\cdot)$ admits a holomorphic continuation to $\rho(\tilde{A})$.*

Proof. By Lemma 4.1, the perturbation determinant $\Delta_{\tilde{A}'/\tilde{A}}^{\Pi}(\cdot)$ is well defined and holomorphic on $\rho(\tilde{A}) \cap \rho(A_0)$. Since $A_0 = A_0^*$ the determinant $\Delta_{\tilde{A}'/\tilde{A}}^{\Pi}(\cdot)$ is holomorphic on $\rho(\tilde{A}) \cap \mathbb{C}_\pm$.

Further, let us assume that $\lambda \in \rho(\tilde{A}) \cap \mathbb{R}$. By Lemma 4.1(i), the set $\rho(\tilde{A}') \cap \rho(\tilde{A})$ is not empty and (1.6) holds. By Proposition 3.8, the pair $\{\tilde{A}', \tilde{A}\}$ is jointly almost solvable with respect to λ . It follows from Proposition 3.3(ii) that there exists a regular boundary triplet $\Pi_\lambda = \{\mathcal{H}^\lambda, \Gamma_0^\lambda, \Gamma_1^\lambda\}$ such that $\lambda \in \rho(A_{0,\lambda})$, $A_{0,\lambda} := A^* \upharpoonright \ker(\Gamma_0^\lambda)$. According to Lemma 4.1(ii) the perturbation determinant $\Delta_{\tilde{A}'/\tilde{A}}^{\Pi_\lambda}(\cdot)$ is holomorphic on $\rho(\tilde{A}) \cap \rho(A_{0,\lambda})$. If $\lambda' \in \rho(\tilde{A}) \cap \mathbb{R}$, we find a regular boundary triplet $\Pi_{\lambda'}$ such that $\lambda' \in \rho(A_{0,\lambda'})$. By Lemma 4.1(ii), the perturbation determinant $\Delta_{\tilde{A}'/\tilde{A}}^{\Pi_{\lambda'}}(z)$ is holomorphic on $\rho(\tilde{A}) \cap \rho(A_{0,\lambda'})$. By Proposition 4.5, there is a constant $c \in \mathbb{C}$ such that $\Delta_{\tilde{A}'/\tilde{A}}^{\Pi_\lambda}(z) = c \Delta_{\tilde{A}'/\tilde{A}}^{\Pi_{\lambda'}}(z)$ for $z \in \rho(\tilde{A}) \cap \mathbb{C}_\pm$. Since the right-hand side of this identity is holomorphic on $\rho(\tilde{A}) \cap \rho(A_{0,\lambda'})$ the left-hand side admits a holomorphic continuation to $\rho(\tilde{A}) \cap \rho(A_{0,\lambda'})$. Since $\lambda' \in \rho(\tilde{A})$ is arbitrary, the proof is complete. \square

Proposition 4.6 enables us to consider the perturbation determinant as a holomorphic function on $\rho(\tilde{A})$. Doing so we immediately obtain the following improvement of Proposition 4.5.

Corollary 4.7. *Let the assumptions of Proposition 4.5 be satisfied. If the extension \tilde{A} is almost solvable, then there exists a constant $c \in \mathbb{C}$ such that $\Delta_{\tilde{A}'/\tilde{A}}^\Pi(z) = c\Delta_{\tilde{A}'/\tilde{A}}^{\Pi'}(z)$ for $z \in \rho(\tilde{A})$.*

Proof. From Proposition 4.6 we immediately get that the relation (4.3) extends to $\rho(\tilde{A})$. \square

4.5 Properties

Let us show that the perturbation determinant $\Delta_{\tilde{A}'/\tilde{A}}^\Pi(\cdot)$ fulfills properties similar to those of Krein's determinant mentioned in Section 2.4.

Proposition 4.8. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* , $M(\cdot)$ the corresponding Weyl function. Assume that the proper extensions $\tilde{A}, \tilde{A}', \tilde{A}'' (\in \text{Ext}_A)$ are disjoint with A_0 , i.e. $\tilde{A} = A_B, \tilde{A}' = A_{B'}, \tilde{A}'' = A_{B''}$ with $B, B', B'' \in \mathcal{C}(\mathcal{H})$.*

(i) *If $B', B \in \mathfrak{S}_1(\mathcal{H})$, then $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$ and*

$$\Delta_{\tilde{A}'/\tilde{A}}^\Pi(z) = \frac{\det(I_{\mathcal{H}} - B'M(z)^{-1})}{\det(I_{\mathcal{H}} - BM(z)^{-1})}, \quad z \in \rho(\tilde{A}) \cap \rho(A_1), \quad (4.8)$$

where $A_1 := A^* \upharpoonright \ker(\Gamma_1)$.

(ii) *Let $\rho(\tilde{A}') \cap \rho(\tilde{A}) \neq \emptyset$. If $\{\tilde{A}'', \tilde{A}'\} \in \mathfrak{D}^\Pi$ and $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$, then $\{\tilde{A}'', \tilde{A}\} \in \mathfrak{D}^\Pi$ and*

$$\Delta_{\tilde{A}''/\tilde{A}'}^\Pi(z) \Delta_{\tilde{A}'/\tilde{A}}^\Pi(z) = \Delta_{\tilde{A}''/\tilde{A}}^\Pi(z), \quad z \in \rho(\tilde{A}') \cap \rho(\tilde{A}). \quad (4.9)$$

(iii) *Let $\rho(\tilde{A}') \cap \rho(\tilde{A}) \neq \emptyset$. If $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$, then $\{\tilde{A}, \tilde{A}'\} \in \mathfrak{D}^\Pi$ and*

$$\Delta_{\tilde{A}'/\tilde{A}}^\Pi(z) \Delta_{\tilde{A}/\tilde{A}'}^\Pi(z) = 1, \quad z \in \rho(\tilde{A}') \cap \rho(\tilde{A}). \quad (4.10)$$

(iv) *Let $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$. If $z_0 \in \rho(A_0)$ is either a regular point or a normal eigenvalue of \tilde{A}' and \tilde{A} , then*

$$\text{ord} \left(\Delta_{\tilde{A}'/\tilde{A}}^\Pi(z_0) \right) = m_{z_0}(\tilde{A}') - m_{z_0}(\tilde{A}). \quad (4.11)$$

In particular, if $z_0 \in \rho(A_0) \cap \rho(\tilde{A})$, then

$$\text{ord} \left(\Delta_{\tilde{A}'/\tilde{A}}^\Pi(z_0) \right) = m_{z_0}(\tilde{A}'), \quad (4.12)$$

cf. Appendix B.

(v) *If $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$, then*

$$\frac{1}{\Delta_{\tilde{A}'/\tilde{A}}^\Pi(z)} \frac{d}{dz} \Delta_{\tilde{A}'/\tilde{A}}^\Pi(z) = \text{tr}((\tilde{A} - z)^{-1} - (\tilde{A}' - z)^{-1}) \quad (4.13)$$

for $z \in \rho(\tilde{A}') \cap \rho(\tilde{A})$, where the right hand side has sense by Lemma 4.1(i).

(vi) If $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$ and $\{\tilde{A}'^*, \tilde{A}^*\} \in \mathfrak{D}^\Pi$, then

$$\Delta_{\tilde{A}'^*/\tilde{A}^*}^\Pi(z) = \overline{\Delta_{\tilde{A}'/\tilde{A}}^\Pi(\bar{z})}, \quad z \in \rho(\tilde{A}^*). \quad (4.14)$$

(vii) If $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$, then

$$\frac{\Delta_{\tilde{A}'/\tilde{A}}^\Pi(z)}{\Delta_{\tilde{A}'/\tilde{A}}^\Pi(\zeta)} = \det(I_{\mathcal{H}} + (M(z) - M(\zeta))(B - M(z))^{-1}(B' - B)(B' - M(\zeta))^{-1}),$$

for $z \in \rho(\tilde{A})$ and $\zeta \in \rho(\tilde{A}') \cap \rho(\tilde{A})$.

Proof. (i) Obviously we have $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$. Further, let $z \in \rho(A_0)$. One has $0 \in \rho(B - M(z))$ if and only if $1 \in \rho(BM(z)^{-1})$. Hence

$$\begin{aligned} I_{\mathcal{H}} + (B' - B)(B - M(z))^{-1} &= (B' - M(z))(B - M(z))^{-1} \\ &= (I_{\mathcal{H}} - B'M(z)^{-1})(I_{\mathcal{H}} - BM(z)^{-1})^{-1}, \quad z \in \rho(\tilde{A}) \cap \rho(A_0). \end{aligned}$$

Since $z \in \rho(A_1)$ if and only if $0 \in \rho(M(z))$ the right-hand side of (4.8) admits a holomorphic continuation to $\rho(\tilde{A}) \cap \rho(A_1)$.

(ii) If $z \in \rho(\tilde{A}') \cap \rho(\tilde{A}) \cap \rho(A_0) \neq \emptyset$, then

$$\{z \in \rho(A_0) : 0 \in \rho(B' - M(z)) \wedge 0 \in \rho(B - M(z))\} \neq \emptyset.$$

Hence $\{z \in \rho(A_0) : 0 \in \rho(B' - M(z))\} \neq \emptyset$. To check Definition 1.1(ii) we note that $\text{dom}(B'') = \text{dom}(B') = \text{dom}(B)$. Using

$$(B'' - B)(B - M(z))^{-1} = (B'' - B')(B - M(z))^{-1} + (B' - B)(B - M(z))^{-1}$$

for $z \in \rho(\tilde{A}') \cap \rho(A_0)$. Notice that $(B' - B)(B - M(z))^{-1} \in \mathfrak{S}_1(\mathcal{H})$ by assumption. It remains to verify that $(B'' - B')(B - M(z))^{-1} \in \mathfrak{S}_1(\mathcal{H})$ for $z \in \rho(\tilde{A}) \cap \rho(A_0)$. Obviously, we have

$$\begin{aligned} (B'' - B')(B - M(z))^{-1} &= (B'' - B')(B' - M(z))^{-1} \\ &\quad + (B'' - B')(B' - M(z))^{-1}(B' - B)((B - M(z))^{-1}) \end{aligned}$$

for $z \in \rho(\tilde{A}') \cap \rho(\tilde{A}) \cap \rho(A_0)$. Since $(B'' - B')(B' - M(z))^{-1} \in \mathfrak{S}_1(\mathcal{H})$ for $z \in \rho(\tilde{A}') \cap \rho(A_0)$ and $(B' - B)(B - M(z))^{-1} \in \mathfrak{S}_1(\mathcal{H})$ for $z \in \rho(\tilde{A}) \cap \rho(A_0)$ by assumption we find $(B'' - B)(B - M(z))^{-1} \in \mathfrak{S}_1(\mathcal{H})$ for $z \in \rho(\tilde{A}') \cap \rho(\tilde{A}) \cap \rho(A_0)$. Now (4.9) follows immediately from the definition for $z \in \rho(\tilde{A}') \cap \rho(\tilde{A}) \cap \rho(A_0)$. Finally, using Proposition 4.6 we can omit $\rho(A_0)$.

(iii) If $\rho(\tilde{A}') \cap \rho(\tilde{A}) \neq \emptyset$, then $\rho(\tilde{A}') \cap \rho(\tilde{A}) \cap \rho(A_0) \neq \emptyset$. Hence one has $\{z \in \rho(A_0) : 0 \in \rho(B' - M(z)) \wedge 0 \in \rho(B - M(z))\} \neq \emptyset$. Therefore the conditions (i) and (ii) of Definition 1.1 are satisfied for the ordered pair $\{\tilde{A}, \tilde{A}'\}$. To prove condition (iii) of Definition 1.1 we use the representation

$$(B - B')(B' - M(z))^{-1} = -(B' - B)(B - M(z))^{-1}(B - M(z))(B' - M(z))^{-1}$$

for $z \in \rho(\tilde{A}') \cap \rho(\tilde{A}) \cap \rho(A_0)$. Since $(B - M(z))(B' - M(z))^{-1}$ is a bounded operator we get $(B - B')(B' - M(z))^{-1} \in \mathfrak{S}_1(\mathcal{H})$ for $z \in \rho(\tilde{A}') \cap \rho(A_0)$. To prove (4.10) it suffices to set $\tilde{A}'' = \tilde{A}$ in (4.9).

(iv) From Proposition 2.5 and the Krein-type formula (2.7) we find that $\nu_{z_0}(\tilde{A}') = \nu_{z_0}(B' - M(z_0))$ and $\nu_{z_0}(\tilde{A}) = \nu_{z_0}(B - M(z_0))$. Following the reasoning of [32, Chapter IV, Sec. 3.4] we get $\text{ord}(\Delta_{\tilde{A}'/\tilde{A}}^\Pi(z_0)) = \nu_{z_0}(B' - M(z_0)) - \nu_{z_0}(B - M(z_0))$ which proves (4.11).

(v) From formula [65, (1.7.10)] we get

$$\begin{aligned} & \frac{1}{\Delta_{\tilde{A}'/\tilde{A}}^\Pi(z)} \frac{d}{dz} \Delta_{\tilde{A}'/\tilde{A}}^\Pi(z) \\ &= \text{tr} \left((I + (B' - B)(B - M(z))^{-1})^{-1} \frac{d}{dz} (B' - B)(B - M(z))^{-1} \right) \end{aligned} \quad (4.15)$$

for $z \in \rho(\tilde{A}') \cap \rho(\tilde{A}) \cap \rho(A_0)$. From (2.6) we find

$$\frac{d}{dz} (B - M(z))^{-1} = (B - M(z))^{-1} \gamma(\bar{z})^* \gamma(z) (B - M(z))^{-1}$$

for $z \in \rho(\tilde{A}) \cap \rho(A_0)$. Combining two last formulas we obtain

$$\begin{aligned} & \text{tr} \left((I + (B' - B)(B - M(z))^{-1})^{-1} \frac{d}{dz} (B' - B)(B - M(z))^{-1} \right) \\ &= \text{tr} (\gamma(z)(B' - M(z))^{-1} (B' - B)(B - M(z))^{-1} \gamma(\bar{z})^*) \end{aligned}$$

for $z \in \rho(\tilde{A}') \cap \rho(\tilde{A}) \cap \rho(A_0)$. On the other hand, the Krein-type formula (2.7) yields

$$(\tilde{A}' - z)^{-1} - (\tilde{A} - z)^{-1} = \gamma(z)(B' - M(z))^{-1} (B - B')(B - M(z))^{-1} \gamma(\bar{z})^*$$

for $z \in \rho(\tilde{A}') \cap \rho(\tilde{A}) \cap \rho(A_0)$. Combining this formula with (4.15) we arrive at (4.13).

(vi) If $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$, then $C(B - M(z))^{-1} \in \mathfrak{S}_1(\mathcal{H})$ for $z \in \rho(\tilde{A}) \cap \rho(A_0)$ where $C := B' - B$, $\text{dom}(C) := \text{dom}(B') = \text{dom}(B)$. Since $\{\tilde{A}'^*, \tilde{A}^*\} \in \mathfrak{D}^\Pi$ the operators B' and B are densely defined. Hence B'^* , B^* and C^* exist and $\overline{(B^* - M(z)^*)^{-1} C^*} \in \mathfrak{S}_1(\mathcal{H})$. Setting $C_* := B'^* - B^*$, $\text{dom}(C_*) = \text{dom}(B'^*) = \text{dom}(B^*)$ we get $\text{dom}(C^*) \supseteq \text{dom}(C_*)$. Moreover, we have

$$C^*(B^* - M(z)^*)^{-1} = C_*(B^* - M(z)^*)^{-1} \in \mathfrak{S}_1(\mathcal{H})$$

for $z \in \rho(\tilde{A}) \cap \rho(A_0)$. Applying Corollary A.4 we obtain

$$\begin{aligned} & \det(I_{\mathcal{H}} + \overline{(B^* - M(z)^*)^{-1} C^*}) \\ &= \det(I_{\mathcal{H}} + C^*(B^* - M(z)^*)^{-1}) = \det(I_{\mathcal{H}} + C_*(B^* - M(z)^*)^{-1}) \end{aligned}$$

for $z \in \rho(\tilde{A}) \cap \rho(A_0)$. Since $((B' - B)(B - M(z))^{-1})^* = \overline{(B^* - M(z)^*)^{-1}C^*}$ we find

$$\begin{aligned} \overline{\Delta_{\tilde{A}'/\tilde{A}}^\Pi(z)} &= \det(I_{\mathcal{H}} + \overline{(B^* - M(z)^*)^{-1}C^*}) \\ &= \det(I_{\mathcal{H}} + C_*(B^* - M(z)^*)^{-1}) = \Delta_{\tilde{A}^*/\tilde{A}^*}^\Pi(\bar{z}) \end{aligned}$$

for $z \in \rho(\tilde{A}) \cap \rho(A_0)$ where we have used $M(z)^* = M(\bar{z})$ for $z \in \rho(A_0)$. Replacing \bar{z} by z it follows (4.14) for $z \in \rho(\tilde{A}^*) \cap \rho(A_0)$.

(vii) The proof follows from (ii) and (iii). \square

Propositions 4.6 and 4.8 show that the perturbation determinant $\Delta_{\tilde{A}'/\tilde{A}}^\Pi(\cdot)$ has the properties similar to that of the classical perturbation determinant.

Proposition 4.9. *Let $\tilde{A} \in \text{Ext}_A$ and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* such that $\tilde{A} = A_B$, $B \in \mathcal{C}(\mathcal{H})$, and $M(\cdot)$ the corresponding Weyl function. If for some $\zeta \in \rho(\tilde{A}) \cap \rho(A_0)$*

$$(\tilde{A} - \zeta)^{-1} - (A_0 - \zeta)^{-1} \in \mathfrak{S}_1(\mathfrak{H}), \quad A_0 := A^* \upharpoonright \ker(\Gamma_0), \quad (4.16)$$

then the following holds:

- (i) B is discrete and $(B - \mu)^{-1} \in \mathfrak{S}_1(\mathcal{H})$ for any $\mu \in \rho(B)$.
- (ii) There exists a regular for $\{\tilde{A}, A_0\}$ boundary triplet $\tilde{\Pi} = \{\tilde{\mathcal{H}}, \tilde{\Gamma}_0, \tilde{\Gamma}_1\}$ for A^* .
- (iii) If $\tilde{\Pi}$ is a boundary triplet for A^* such that $\{\tilde{A}, A_0\} \in \mathfrak{D}^{\tilde{\Pi}}$, then there exist $\mu = \bar{\mu} \in \rho(B)$ and $c \in \mathbb{C}$ such that

$$\Delta_{\tilde{A}/A_0}^{\tilde{\Pi}}(z) = c \det(I - (\mu - B)^{-1}(\mu - M(z))), \quad z \in \rho(A_0). \quad (4.17)$$

If $0 \in \rho(B)$, then one can put $\mu = 0$ in (4.17).

Proof. (i) If (4.16) is satisfied, then, by Proposition 2.6(i), B is unbounded, its spectrum is discrete and $(B - \mu)^{-1} \in \mathfrak{S}_1(\mathcal{H})$ for $\mu \in \rho(B)$.

(ii) Since $A_0 = A_0^*$, Proposition 3.5 yields that A_0 is almost solvable. By Proposition 4.3, there exists a boundary triplet $\tilde{\Pi}$ for A^* which is regular for the pair $\{\tilde{A}, A_0\}$, in particular, $\{\tilde{A}, A_0\} \in \mathfrak{D}^{\tilde{\Pi}}$.

(iii) Since B is discrete, $\rho(B) \cap \mathbb{R} \neq \emptyset$. Choosing $\mu \in \rho(B) \cap \mathbb{R}$ we set $\Gamma'_1 := \Gamma_0$ and $\Gamma'_0 := -(\Gamma_1 + \mu\Gamma_0)$ and note that $\Pi' = \{\mathcal{H}, \Gamma'_0, \Gamma'_1\}$ is a boundary triplet for A^* too. Moreover, we have $\tilde{A} = A_{\tilde{B}}$ where $\tilde{B} = (\mu - B)^{-1}$ and $A_0 = A_0 = A^* \upharpoonright \ker(\tilde{\Gamma}_1)$ where \mathbb{O} is the zero operator in \mathcal{H} . Thus, the boundary triplet $\tilde{\Pi}$ is regular for $\{\tilde{A}, A_0\}$. The Weyl function $\tilde{M}(\cdot)$ associated with $\tilde{\Pi}$ is $\tilde{M}(z) = (\mu - M(z))^{-1}$, $z \in \mathbb{C}_\pm$. Hence, by Definition 1.1, the perturbation determinant is

$$\Delta_{\tilde{A}/A_0}^{\tilde{\Pi}}(z) = \det(I - \tilde{B}\tilde{M}^{-1}(z)) = \det(I - \tilde{B}(\mu - M(z))), \quad z \in \mathbb{C}_\pm. \quad (4.18)$$

Finally, applying Proposition 4.5 we arrive at (4.17). \square

Corollary 4.10. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* , $M(\cdot)$ the corresponding Weyl function and let $\tilde{A}', \tilde{A} \in \text{Ext}_A$ be such that $\tilde{A}' = A_{B'}$, $\tilde{A} = A_B$, where $B', B \in \mathcal{C}(\mathcal{H})$. If $\rho(\tilde{A}') \cap \rho(\tilde{A}) \neq \emptyset$ and*

$$(\tilde{A}' - \zeta)^{-1} - (A_0 - \zeta)^{-1} \in \mathfrak{S}_1(\mathfrak{H}) \quad \text{and} \quad (\tilde{A} - \zeta)^{-1} - (A_0 - \zeta)^{-1} \in \mathfrak{S}_1(\mathfrak{H}) \quad (4.19)$$

for some $\zeta \in \rho(\tilde{A}') \cap \rho(\tilde{A}) \cap \mathbb{C}_\pm$, then the following holds:

- (i) B' and B are discrete and there exists $\mu \in \rho(B') \cap \rho(B) \cap \mathbb{R}$ such that $(B' - \mu)^{-1} \in \mathfrak{S}_1(\mathcal{H})$ and $(B - \mu)^{-1} \in \mathfrak{S}_1(\mathcal{H})$.
- (ii) There exists a boundary triplet $\tilde{\Pi} = \{\tilde{\mathcal{H}}, \tilde{\Gamma}_0, \tilde{\Gamma}_1\}$ for A^* , which can be chosen to be regular for the family $\{\tilde{A}', \tilde{A}, A_0\}$.
- (iii) If $\tilde{\Pi} = \{\tilde{\mathcal{H}}, \tilde{\Gamma}_0, \tilde{\Gamma}_1\}$ is a (not necessarily regular for $\{\tilde{A}', \tilde{A}, A_0\}$) boundary triplet for A^* such that $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^{\tilde{\Pi}}$, $\{\tilde{A}', A_0\} \in \mathfrak{D}^{\tilde{\Pi}}$ and $\{\tilde{A}, A_0\} \in \mathfrak{D}^{\tilde{\Pi}}$, then the perturbation determinant $\Delta_{\tilde{A}'/\tilde{A}}^{\tilde{\Pi}}(\cdot)$ admits the representation

$$\Delta_{\tilde{A}'/\tilde{A}}^{\tilde{\Pi}}(z) = c \frac{\det(I - (\mu - B')^{-1}(\mu - M(z)))}{\det(I - (\mu - B)^{-1}(\mu - M(z)))}, \quad z \in \rho(\tilde{A}) \cap \mathbb{C}_\pm. \quad (4.20)$$

If $0 \in \rho(B') \cap \rho(B)$, then we can put $\mu = 0$ in (4.20).

Proof. (i) This statement follows immediately from Proposition 4.9(i).

(ii) Since $A_0 = A_0^*$, it is almost solvable. Therefore, it follows from Proposition 3.7 and (4.19) that the system $\{\tilde{A}', \tilde{A}, A_0\}$ is jointly almost solvable. By Proposition 3.3(i), there exists a regular boundary triplet $\tilde{\Pi}$ for $\{\tilde{A}', \tilde{A}, A_0\}$. Finally, Proposition 2.6 yields the inclusions $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^{\tilde{\Pi}}$, $\{\tilde{A}', A_0\} \in \mathfrak{D}^{\tilde{\Pi}}$ and $\{\tilde{A}, A_0\} \in \mathfrak{D}^{\tilde{\Pi}}$.

(iii) As in the proof of Proposition 4.9 we introduce the boundary triplet $\tilde{\Pi} = \{\mathcal{H}, \tilde{\Gamma}_0, \tilde{\Gamma}_1\}$ where $\tilde{\Gamma}_1 := \Gamma_0$ and $\tilde{\Gamma}_0 := -(\Gamma_1 + \mu\Gamma_0)$. Clearly, $\tilde{A}' = A_{\tilde{B}'}$, $\tilde{B}' := (\mu - B')^{-1}$ and $\tilde{A} = A_{\tilde{B}}$, $\tilde{B} = (\mu - B)^{-1}$ and $A_0 = A_{\mathbb{O}}$ where \mathbb{O} is the zero operator. Hence the boundary triplet $\tilde{\Pi}$ is regular for the set of operators $\{\tilde{A}', \tilde{A}, A_0\}$. Combining the chain rule (cf. (4.9)) with Proposition 4.9(iii) we arrive at (4.20). To complete the proof it remains to apply Proposition 4.5. \square

5 Perturbation determinants and trace formulas

5.1 Pairs of selfadjoint extensions

A spectral shift function has originally been introduced by I.M. Lifshitz in a special case and by M.G. Krein [40] in the general case. Namely, Krein [40] proved that for a pair $\{H' = H + V, H\}$ of selfadjoint operators with $V \in \mathfrak{S}_1(\mathfrak{H})$ there exists a unique real-valued function $\xi(\cdot) \in L^1(\mathbb{R})$ such that the following trace formula holds

$$\text{tr}((H' - z)^{-1} - (H - z)^{-1}) = - \int_{\mathbb{R}} \frac{\xi(t)}{(t - z)^2} dt, \quad z \in \rho(H') \cap \rho(H). \quad (5.1)$$

Formula (5.1) has been extended in [44] to a pair of selfadjoint operators $\{H', H\}$ which are only resolvent comparable, that is, $(H - \zeta)^{-1} - (H_0 - \zeta)^{-1} \in \mathfrak{S}_1(\mathfrak{H})$ for some $\zeta \in \rho(H') \cap \rho(H)$. In this case formula (5.1) remains valid. However, one has only $\xi(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ which yields that the spectral shift function $\xi(\cdot)$ is not uniquely defined by (5.1): alongside with $\xi(\cdot)$ any function $\xi(\cdot) + c$, $c \in \mathbb{R}$, satisfies (5.1) too. First we show that the converse is also true.

Lemma 5.1. *Let H and H_0 be selfadjoint operators which are resolvent comparable. Assume that there exist real-valued functions $\tilde{\xi}(\cdot), \xi(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ such that the trace formula (5.1) holds with both $\xi(\cdot)$ and $\tilde{\xi}(\cdot)$. Then $\tilde{\xi}(t) - \xi(t) = c$ for a.e. $t \in \mathbb{R}$ where c is a real constant.*

Proof. Let $\eta(t) := \tilde{\xi}(t) - \xi(t)$, $t \in \mathbb{R}$. Then

$$\int_{-\infty}^{\infty} \frac{\eta(t) dt}{(t-z)^2} = 0, \quad z \in \mathbb{C}_+ \cup \mathbb{C}_-, \quad (5.2)$$

and $\eta(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$. We set

$$\mathcal{P}_\eta(z, \bar{z}) := \frac{1}{\pi} \int \frac{y \eta(t) dt}{|t-z|^2} = \frac{1}{2i\pi} \int \left(\frac{1}{t-z} - \frac{1}{t-\bar{z}} \right) \eta(t) dt, \quad (5.3)$$

where $z = x + iy \in \mathbb{C}_\pm$. Differentiating $\mathcal{P}_\eta(z, \bar{z})$ with respect to z and \bar{z} and taking (5.2) into account we get

$$\frac{\partial}{\partial z} \mathcal{P}_\eta(z, \bar{z}) = \frac{\partial}{\partial \bar{z}} \mathcal{P}_\eta(z, \bar{z}) = 0.$$

Thus, $\mathcal{P}_\eta(z, \bar{z})$ is holomorphic and anti-holomorphic in $\mathbb{C}_+ \cup \mathbb{C}_-$. Hence $\mathcal{P}_\eta(z, \bar{z}) = a = \text{const.}$, $z \in \mathbb{C}_+$. Applying the Fatou theorem to (5.3) we get

$$\eta(t) = \mathcal{P}(t + i0, t - i0) = a = \bar{a} = \text{const.}$$

for a.e. $t \in \mathbb{R}$. □

In the sequel we need the following technical lemma.

Lemma 5.2. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* and $M(\cdot)$ the corresponding Weyl function. Let also B be a maximal accumulative operator, in particular, a selfadjoint operator. Then the following statements are true:*

(i) *If $V_+ \in \mathfrak{S}_1(\mathcal{H})$ and $V_+ \geq 0$, then there exist a constant $c_+ > 0$ and a non-negative function $\xi_+(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ such that the following representation holds*

$$\det(I + V_+(B - M(z))^{-1}) = c_+ \exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \xi_+(t) dt \right\}, \quad (5.4)$$

$z \in \mathbb{C}_+$.

(ii) If $V = V^* \in \mathfrak{S}_1(\mathcal{H})$, then there exist a constant $c > 0$ and a real-valued function $\xi(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ such that the representation

$$\det(I + V(B - M(z))^{-1}) = c \exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \xi(t) dt \right\} \quad (5.5)$$

holds for $z \in \mathbb{C}_+$.

Proof. (i) We introduce the operator-valued Nevanlinna function

$$\Omega_+(z) := I + \sqrt{V_+}(B - M(z))^{-1} \sqrt{V_+}, \quad z \in \mathbb{C}_+.$$

Since $\Omega_+(z)$ is m -dissipative for $z \in \mathbb{C}_+$ and $0 \in \rho(\Omega_+(z))$, $z \in \mathbb{C}_+$, the operator-valued function $\log(\Omega_+(z))$ is well-defined by (C.2) for $z \in \mathbb{C}_+$. Since $\log(\Omega_+(z)) \in \mathfrak{S}_1(\mathcal{H})$ [26, Theorem 2.8] guarantees the existence of a measurable function $\Xi_+(\cdot) : \mathbb{R} \rightarrow \mathfrak{S}_1(\mathcal{H})$ such that $\Xi(t) \geq 0$ for a.e. $t \in \mathbb{R}$ and $\text{tr}(\Xi(\cdot)) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$. Moreover, the following representation holds

$$\log(\Omega_+(z)) = \Omega_+ + \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \Xi_+(t) dt, \quad z \in \mathbb{C}_+,$$

where the integral is taken in the weak sense and $\Omega_+ = \Omega_+^* \in \mathfrak{S}_1(\mathcal{H})$. Setting $\xi_+(t) := \text{tr}(\Xi_+(t))$, $t \in \mathbb{R}$, we define a non-negative function $\xi_+(\cdot)$ satisfying $\xi_+(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ and such that

$$\text{tr}(\log(\Omega_+(z))) = \text{tr}(\Omega_+) + \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \xi_+(t) dt, \quad z \in \mathbb{C}_+.$$

Taking into account (C.3) we verify (5.4) with $c_+ := \exp\{\text{tr}(\Omega_+)\} > 0$.

(ii) Using the decomposition $V = V_+ - V_-$, $V_{\pm} \geq 0$, we set $B_- := B - V_-$. It follows from the identity

$$(I + V(B - M(z))^{-1})(I + V_-(B_- - M(z))^{-1}) = I + V_+(B_- - M(z))^{-1}$$

that

$$\det(I + V(B - M(z))^{-1}) = \frac{\det(I + V_+(B_- - M(z))^{-1})}{\det(I + V_-(B_- - M(z))^{-1})}, \quad z \in \mathbb{C}_+. \quad (5.6)$$

Combining (5.6) with the representation (5.4) we arrive at (5.5). \square

Lemma 5.2 implies the following representation theorem for a perturbation determinant.

Theorem 5.3. *Let $\tilde{A}', \tilde{A} \in \text{Ext}_A$ be selfadjoint extensions of A such that the pair $\{\tilde{A}', \tilde{A}\}$ is resolvent comparable, that is, condition (1.6) is satisfied. Then the following holds:*

(i) *There exists a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* , which can be chosen regular for $\{\tilde{A}', \tilde{A}\}$, such that $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$.*

(ii) If $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$, then there exist a real-valued function $\xi(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2}dt)$ and a constant $c = \bar{c}$ such that the following representation

$$\Delta_{\tilde{A}'/\tilde{A}}^\Pi(z) = c \exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \xi(t) dt \right\}, \quad z \in \mathbb{C}_\pm. \quad (5.7)$$

holds. Moreover, there exists an integer $n \in \mathbb{Z}$ such that

$$\xi(t) = \lim_{\varepsilon \rightarrow +0} \operatorname{Im} (\log(\Delta_{\tilde{A}'/\tilde{A}}^\Pi(t + i\varepsilon))) + n\pi, \quad \text{for a.e. } t \in \mathbb{R}. \quad (5.8)$$

(iii) The trace formula

$$\operatorname{tr} \left((\tilde{A}' - z)^{-1} - (\tilde{A} - z)^{-1} \right) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\xi(t)}{(t-z)^2} dt, \quad z \in \mathbb{C}_\pm. \quad (5.9)$$

is valid where $\xi(\cdot)$ is given by (ii). Any real-valued function $\tilde{\xi}(\cdot)$ satisfying (5.9) differs from $\xi(\cdot)$ by an additive real constant.

Proof. (i) This statement follows immediately is from Corollary 4.4(i).

(ii) At first, let us assume that Π is regular for $\{\tilde{A}', \tilde{A}\}$. Further, we assume that $\tilde{A}' = A_{B'}$ and $\tilde{A} = A_B$, where B' and B are bounded selfadjoint operators. By Lemma 5.2(ii) there exist a real constant c and a real-valued function $\xi(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2}dt)$ such that the following representation

$$\begin{aligned} \Delta_{\tilde{A}'/\tilde{A}}^\Pi(z) &= \det (I + (B' - B)(B - M(z))^{-1}) \\ &= c \exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \xi(t) dt \right\} \end{aligned} \quad (5.10)$$

holds for $z \in \mathbb{C}_\pm$. This proves (5.7). If Π is not regular, one obtains (5.7) by combining (5.10) with Proposition 4.5.

It follows from (5.7) that there exists an integer $n \in \mathbb{Z}$ such that

$$\log(\Delta_{\tilde{A}'/\tilde{A}}^\Pi(z)) = \log(|c|) + \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \xi(t) dt + in\pi, \quad (5.11)$$

for $z \in \mathbb{C}_+$. At first we find that this representation is true in a neighborhood of a point $z \in \mathbb{C}_+$ such that the value $\Delta_{\tilde{A}'/\tilde{A}}^\Pi(z)$ does not belong to the negative imaginary semi-axis. By analytical continuation it holds for $z \in \mathbb{C}_+$. Taking the imaginary part of both sides of (5.11) and applying Fatou's theorem we arrive at (5.8).

(iii) From (ii) and Proposition 4.8(v) we immediately obtain that the trace formula (5.9) holds choosing $\tilde{\xi}(t) := \xi(t)$, $t \in \mathbb{R}$. Applying Lemma 5.1 we get that any real-valued function $\tilde{\xi}(\cdot)$ obeying (5.9) differs from $\xi(\cdot)$ by a real constant. \square

Any function $\tilde{\xi}(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2}dt)$ obeying (5.9) is called the spectral shift function of the pair $\{\tilde{A}', \tilde{A}\}$ of extensions of A . Theorem 5.3(iii) shows that each pair $\{\tilde{A}', \tilde{A}\}$ admits many spectral shift functions.

Theorem 5.4. *Let $\tilde{A}_2, \tilde{A}_1, H \in \text{Ext}_A$ be selfadjoint extensions of A . Assume that the pairs $\{\tilde{A}_1, H\}$ and $\{\tilde{A}_2, H\}$ are resolvent comparable, that is, conditions of type (1.6) are satisfied. If for some $\lambda_0 \in \rho(\tilde{A}_2) \cap \rho(\tilde{A}_1) \cap \mathbb{R}$ the condition*

$$(\tilde{A}_2 - \lambda_0)^{-1} \leq (\tilde{A}_1 - \lambda_0)^{-1} \quad (5.12)$$

is valid, then the real-valued spectral shift functions $\xi_1(\cdot)$ and $\xi_2(\cdot)$ of the pairs $\{\tilde{A}_1, H\}$ and $\{\tilde{A}_2, H\}$, respectively, can be chosen such that $\xi_1(t) \leq \xi_2(t)$ holds for a.e. $t \in \mathbb{R}$.

Proof. By Corollary 4.4(ii) there is a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* , which is regular for $\{\tilde{A}', \tilde{A}, H\}$ and satisfies $\lambda \in \rho(A_0)$, such that $\{\tilde{A}', H\} \in \mathfrak{D}^\Pi$, $\{\tilde{A}, H\} \in \mathfrak{D}^\Pi$ and $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$. By Theorem 5.3(ii) there are real-valued functions $\xi_j(\cdot) \in L^2(\mathbb{R}, \frac{1}{1+t^2} dt)$, $j = 1, 2$, such that the perturbations determinants $\Delta_{\tilde{A}_1/H}^\Pi(z)$ admits the representations

$$\Delta_{\tilde{A}_1/H}^\Pi(z) = c_1 \exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \xi_1(t) dt \right\}, \quad z \in \mathbb{C}_\pm, \quad (5.13)$$

$c_1 \in \mathbb{R}$, are valid. Taking into account the chain rule, cf. Proposition 4.8(ii), we get

$$\Delta_{\tilde{A}_2/H}^\Pi(z) = \Delta_{\tilde{A}_2/\tilde{A}_1}^\Pi(z) \Delta_{\tilde{A}_1/H}^\Pi(z), \quad z \in \mathbb{C}_\pm. \quad (5.14)$$

Assume that $\tilde{A}_j = A_{B_j}$ where B_j are bounded operators in \mathcal{H} . Since $\lambda_0 \in \rho(\tilde{A}_2) \cap \rho(\tilde{A}_1) \cap \rho(A_0)$, Proposition 2.5(i) yields $0 \in \rho(B_2 - M(\lambda_0))$ and $0 \in \rho(B_1 - M(\lambda_0))$. Hence

$$\begin{aligned} 0 &\leq (\tilde{A}_1 - \lambda_0)^{-1} - (\tilde{A}_2 - \lambda_0)^{-1} \\ &= \gamma(\lambda_0) ((B_1 - M(\lambda_0))^{-1} - (B_2 - M(\lambda_0))^{-1}) \gamma(\lambda_0)^*. \end{aligned}$$

It follows from (5.12) that $(B_1 - M(\lambda_0))^{-1} - (B_2 - M(\lambda_0))^{-1} \geq 0$. Next we introduce a new boundary triplet $\tilde{\Pi} := \{\mathcal{H}, \tilde{\Gamma}_0, \tilde{\Gamma}_1\}$ for A^* by setting

$$\tilde{\Gamma}_1 := -\Gamma_0, \quad \tilde{\Gamma}_0 := \Gamma_1 - M(\lambda_0)\Gamma_0. \quad (5.15)$$

Clearly, $\tilde{A}_j = A_{\tilde{B}_j} = A^* \upharpoonright \ker(\tilde{\Gamma}_1 - \tilde{B}_j \tilde{\Gamma}_0)$, where $\tilde{B}_j := -(B_j - M(\lambda_0))^{-1}$, $j = 1, 2$. Notice that $\{\tilde{A}_2, \tilde{A}_1\} \in \mathfrak{D}^{\tilde{\Pi}}$. By Definition 1.1 one has

$$\Delta_{\tilde{A}_2/\tilde{A}_1}^{\tilde{\Pi}}(z) = \det(I + (\tilde{B}_2 - \tilde{B}_1)(\tilde{B}_1 - \tilde{M}(z))^{-1}), \quad z \in \mathbb{C}_\pm,$$

where $\tilde{M}(\cdot)$ is the Weyl function corresponding to the boundary triplet $\tilde{\Pi}$. Since $\tilde{B}_2 - \tilde{B}_1 \geq 0$, Lemma 5.2(i) implies existence of a non-negative function $\xi(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ such that the representation

$$\Delta_{\tilde{A}_2/\tilde{A}_1}^{\tilde{\Pi}}(z) = \tilde{c} \exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \xi(t) dt \right\}, \quad z \in \mathbb{C}_+, \quad (5.16)$$

$\tilde{c} > 0$, holds. By Proposition 4.5 there is a real constant \tilde{c}_{21} such that $\Delta_{\tilde{A}_2/\tilde{A}_1}^\Pi(z) = \tilde{c}_{21} \Delta_{\tilde{A}_2/\tilde{A}_1}^\Pi(z)$, $z \in \mathbb{C}_\pm$, which yields

$$\Delta_{\tilde{A}_2/\tilde{A}_1}^\Pi(z) = c_{21} \exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \xi(t) dt \right\}, \quad z \in \mathbb{C}_\pm, \quad (5.17)$$

$c_{21} := \tilde{c}_{21}\tilde{c} \in \mathbb{R}$. Inserting (5.13) and (5.17) into (5.14) we find

$$\Delta_{\tilde{A}_2/H}^\Pi = c_2 \exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \xi_2(t) dt \right\}, \quad z \in \mathbb{C}_\pm, \quad (5.18)$$

$c_2 := c_{21}c_1 \in \mathbb{R}$, $\xi_2(t) := \xi_1(t) + \xi(t)$, $t \in \mathbb{R}$. Using Proposition 4.8(v) one gets from (5.13) and (5.18) that $\xi_1(t)$ and $\xi_2(t)$ are spectral shift functions for the pairs $\{\tilde{A}_1, H\}$ and $\{\tilde{A}_2, H\}$, respectively. Since $\xi(t) \geq 0$ for a.e. $t \in \mathbb{R}$ we obtain $\xi_1(t) \leq \xi_2(t)$ for a.e. $t \in \mathbb{R}$. \square

Corollary 5.5. *Let $\tilde{A}_2, \tilde{A}_1 \in \text{Ext}_A$ be selfadjoint extensions of A . If the pair $\{\tilde{A}_2, \tilde{A}_1\}$ is resolvent comparable and condition (5.12) is satisfied for some $\lambda_0 \in \rho(\tilde{A}_2) \cap \rho(\tilde{A}_1) \cap \mathbb{R}$, then the real-valued spectral shift function $\xi(\cdot)$ of the pair $\{\tilde{A}_2, \tilde{A}_1\}$ can be chosen such that $\xi(t) \geq 0$ for a.e. $t \in \mathbb{R}$.*

Proof. By Corollary 4.4(ii) there is a boundary triplet Π for A^* which is regular for $\{\tilde{A}_2, \tilde{A}_1\}$ and $\lambda_0 \in \rho(A_0)$. We set $B := \frac{B_1+B_2}{2}$ and $H := A_B$. From $B_2 - B_1 \in \mathfrak{S}_1(\mathcal{H})$ we get $B_2 - B \in \mathfrak{S}_1(\mathcal{H})$ and $B_1 - B \in \mathfrak{S}_1(\mathcal{H})$. Hence the pairs $\{\tilde{A}_2, H\}$ and $\{\tilde{A}_1, H\}$ are resolvent comparable. By Theorem 5.4 there are real-valued spectral shift functions $\xi_2(\cdot)$ and $\xi_1(\cdot)$ of the pairs $\{\tilde{A}_2, H\}$ and $\{\tilde{A}_1, H\}$, respectively, satisfying $\xi_1(t) \leq \xi_2(t)$ for a.e. $t \in \mathbb{R}$. Setting $\xi(t) := \xi_2(t) - \xi_1(t) \geq 0$, $t \in \mathbb{R}$, we define a real-valued spectral shift function of the pair $\{\tilde{A}_2, \tilde{A}_1\}$. \square

5.2 Pairs of accumulative extensions

We are going to prove a technical lemma which will be important to prove a new type of trace formula in the following.

Lemma 5.6. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* , $M(\cdot)$ the corresponding Weyl function and let B be a bounded accumulative operator in \mathcal{H} .*

(i) *If $0 \leq V_+ \leq |B_I| = -B_I$, $B_I := \text{Im}(B) \leq 0$ and $V_+ \in \mathfrak{S}_1(\mathcal{H})$, then the function $w_+(\cdot) := \det(I + iV_+(B - M(\cdot))^{-1})$ is holomorphic and contractive in \mathbb{C}_+ . Moreover, there exist a non-negative function $\eta_+(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2}dt)$ and a number $\varkappa_+ \in \mathbb{T}$ such that the following representation holds*

$$w_+(z) = \varkappa_+ \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \eta_+(t) dt \right\}, \quad z \in \mathbb{C}_+, \quad (5.19)$$

where $\eta_+(t) = -\ln(|\det(w_+(t+i0))|)$ for a.e. $t \in \mathbb{R}$, i.e. $w_+(\cdot)$ is an outer function.

(ii) If $V \leq |B_I| = -B_I$ and $V \in \mathfrak{S}_1(\mathcal{H})$, then there exist a real-valued function $\eta(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ and a complex number $\varkappa \in \mathbb{T}$ such that the perturbation determinant $w(\cdot) := \det(I + iV(B - M(\cdot))^{-1})$ admits the following representation

$$w(z) = \varkappa \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \eta(t) dt \right\}, \quad z \in \mathbb{C}_+, \quad (5.20)$$

where $\eta(t) = -\ln(|\det(w(t+i0))|)$ for a.e. $t \in \mathbb{R}$, i.e. $w(\cdot)$ is an outer function.

Proof. Since $V_+ \in \mathfrak{S}_1$, it admits the spectral decomposition $V_+ = \sum_{k \in \mathbb{N}} \mu_k(\cdot, \psi_k) \psi_k$ where $\mu_k \geq 0$, $\{\mu_k\}_{k \in \mathbb{N}} \in l_1$ and $\{\psi_k\}_{k \in \mathbb{N}}$ is an orthonormal system. We set

$$B_0 := B_R + i(B_I + V_+) \quad \text{and} \quad B_l := B_0 - i \sum_{k=1}^l \mu_k(\cdot, \psi_k) \psi_k, \quad l \in \mathbb{N}.$$

Notice that $B_l = B + i \sum_{k=l+1}^{\infty} \mu_k(\cdot, \psi_k) \psi_k$ and $\lim_{l \rightarrow \infty} \|B_l - B\|_{\mathfrak{S}_1} = 0$ where $\|\cdot\|_{\mathfrak{S}_1}$ denotes the trace norm.

By assumption, $B_I + V_+ \leq 0$ the operator B_0 is m -accumulative. Further let us introduce the operator-valued function

$$W_l(z) := I + i\mu_l P_l (B_l - M(z))^{-1} P_l, \quad P_l := (\cdot, \psi_l) \psi_l, \quad z \in \mathbb{C}_+, \quad l \in \mathbb{N}.$$

We set $w_l(z) := \det(W_l(z))$, $z \in \mathbb{C}_+$, $l \in \mathbb{N}$. Clearly,

$$w_l(z) = 1 + i\mu_l ((B_l - M(z))^{-1} \psi_l, \psi_l), \quad z \in \mathbb{C}_+, \quad l \in \mathbb{N}. \quad (5.21)$$

Further, we set

$$\Delta_{B_{l-1}/B_l}(z) := \det(I + (B_{l-1} - B_l)(B_l - M(z))^{-1}), \quad z \in \mathbb{C}_+, \quad l \in \mathbb{N}.$$

Since $B_{l-1} - B_l = i\mu_l(\cdot, \psi_l) \psi_l$, $l \in \mathbb{N}$, we get $\Delta_{B_{l-1}/B_l}(z) = w_l(z)$, $z \in \mathbb{C}_+$, $l \in \mathbb{N}$. Due to the chain rule we obtain

$$\Delta_{B_0/B_l}(z) = \prod_{k=1}^l \Delta_{B_{k-1}/B_k}(z) = \prod_{k=1}^l w_k(z), \quad z \in \mathbb{C}_+, \quad l \in \mathbb{N}. \quad (5.22)$$

Since $B_0 - B = V_+$ we have $\det(W_+(z)) = \Delta_{B_0/B}(z)$. By $\lim_{l \rightarrow \infty} \|B_l - B\|_{\mathfrak{S}_1} = 0$ we get from (5.22) that

$$w_+(z) := \det(W_+(z)) = \Delta_{B_0/B}(z) = \lim_{l \rightarrow \infty} \Delta_{B_0/B_l}(z) = \lim_{l \rightarrow \infty} \prod_{k=1}^l w_k(z) \quad (5.23)$$

for $z \in \mathbb{C}_+$. Note, that alongside with $W_+(\cdot)$, the operator function $W_l(\cdot)$, $l \in \mathbb{N}$, is holomorphic and contractive in \mathbb{C}_+ . Hence $w_l(z) = \det(W_l(z))$, $l \in \mathbb{N}$, is holomorphic and contractive in \mathbb{C}_+ , thus $w_l(\cdot) \in H^\infty(\mathbb{C}_+)$. Next we set

$$\theta_l(z) := \Delta_{B_l/B_{l-1}}(z) := 1 - i\mu_l ((B_{l-1} - M(z))^{-1} \psi_l, \psi_l), \quad z \in \mathbb{C}_+, \quad l \in \mathbb{N}.$$

Notice that $\theta_l(z)$ is well defined since B_{l-1} is accumulative. Moreover, one has

$$\theta_l(z)w_l(z) = w_l(z)\theta_l(z) = 1, \quad z \in \mathbb{C}_+, \quad l \in \mathbb{N}. \quad (5.24)$$

Since B_{l-1} is accumulative, $\operatorname{Im}((B_{l-1} - M(z))^{-1}) > 0$, hence

$$\operatorname{Re}(\theta_l(z)) = 1 + \mu_l \operatorname{Im}(((B_{l-1} - M(z))^{-1}\psi_l, \psi_l)) > 1, \quad z \in \mathbb{C}_+, \quad l \in \mathbb{N}.$$

Combining this inequality with (5.24) we get

$$\operatorname{Re}(w_l(z)) = \frac{1}{|\theta_l(z)|^2} \operatorname{Re}(\theta_l(z)) > \frac{1}{|\theta_l(z)|^2} > 0, \quad z \in \mathbb{C}_+, \quad l \in \mathbb{N}.$$

By [22, Corollary II.4.8], for each $l \in \mathbb{N}$ the function $w_l(z)$ is an outer function. According to (D.6) it admits the representation

$$w_l(z) = \varkappa_l \exp \left\{ \frac{i}{\pi} \int_R \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \eta_l(t) dt \right\}, \quad \varkappa_l \in \mathbb{T},$$

for $z \in \mathbb{C}_+$, $l \in \mathbb{N}$, where $\eta_l(t) := -\ln(|w_l(t+i0)|)$, $t \in \mathbb{R}$. Hence

$$\prod_{k=1}^l w_l(z) = \prod_{k=1}^l \varkappa_k \exp \left\{ \sum_{k=1}^l \frac{i}{\pi} \int_R \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \eta_k(t) dt \right\} \quad (5.25)$$

for $z \in \mathbb{C}_+$ and $l \in \mathbb{N}$. Now (5.22) yields

$$0 \leq |\Delta_{B_0/B_l}(z)| = \exp \left\{ - \sum_{k=1}^l \frac{1}{\pi} \int_R \frac{y}{(t-x)^2 + y^2} \eta_k(t) dt \right\}$$

where $z = x + iy$. Since $w_k(z)$, $z \in \mathbb{C}_+$, is contractive, we get $\eta_k(t) \geq 0$ for a.e. $t \in \mathbb{R}$. Combining Corollary E.2 with (5.21) we obtain

$$- \int_R \ln(|w_k(t+i0)|) \frac{1}{1+t^2} dt \leq 2\pi |w_k(i) - 1| \leq 2\pi \mu_k \frac{1}{\|\operatorname{Im}(M(i))\|}, \quad k \in \mathbb{N}.$$

Since $\{\mu_k\}_{k \in \mathbb{N}} \in l_1$, the Beppo Levi theorem yields

$$0 \leq \eta_+(t) := \sum_{k \in \mathbb{N}} \eta_k(t) = - \sum_{k \in \mathbb{N}} \ln(|w_k(t+i0)|) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt),$$

and

$$\frac{i}{\pi} \int_R \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \eta_+(t) dt = \sum_{k=1}^{\infty} \frac{i}{\pi} \int_R \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \eta_k(t) dt. \quad (5.26)$$

It follows from (5.25) and (5.26) that

$$\begin{aligned} w_+(z) &= \lim_{l \rightarrow \infty} \prod_{k=1}^l w_k(z) = \\ &= \left(\lim_{l \rightarrow \infty} \prod_{k=1}^l \varkappa_k \right) \exp \left\{ \frac{i}{\pi} \int_R \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \eta_+(t) dt \right\}, \quad z \in \mathbb{C}_+, \end{aligned}$$

where $w_+(\cdot) = \det(W_+(\cdot))$ and $W_+(\cdot)$ is given by (5.54). Hence the limit $\varkappa_+ := \lim_{l \rightarrow \infty} \prod_{k=1}^l \varkappa_k \in \mathbb{T}$ exists and we arrive at the representation (5.19). Thus, $w_+(\cdot)$ is the outer function and $\eta_+(t) = -\ln(|\det(w_+(t+i0))|)$ for a.e. $t \in \mathbb{R}$, see Appendix D.

(ii) Let $V = V_+ - V_-$, $V_{\pm} \geq 0$. We set $B_- := B - iV_-$. Since $(B_-)_I = B_I - V_- \leq 0$, the operator B_- is accumulative. According to (5.6)

$$\det(I + iV(B - M(z))^{-1}) = \frac{\det(I + iV_+(B_- - M(z))^{-1})}{\det(I + iV_-(B_- - M(z))^{-1})}, \quad z \in \mathbb{C}_+.$$

The assumption $V \leq -B_I$ yields $0 \leq V_+ \leq -B_I + V_- = -(B_-)_I$. Applying (i) we get the existence of a complex number $\varkappa_+ \in \mathbb{T}$ and a non-negative function $\eta_+(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ such that the representation

$$\det(I + iV_+(B_- - M(z))^{-1}) = \varkappa_+ \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \eta_+(t) dt \right\},$$

is valid for $z \in \mathbb{C}_+$. From $0 \leq V_- \leq -B_I + V_- = (B_-)_I$ and (i) we get the existence of a complex number $\varkappa_- \in \mathbb{T}$ and a non-negative function $\eta_-(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ such that the representation

$$\det(I + iV_-(B_- - M(z))^{-1}) = \varkappa_- \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \eta_-(t) dt \right\},$$

holds for $z \in \mathbb{C}_-$. Setting $\varkappa := \varkappa_+/\varkappa_- \in \mathbb{T}$ and $\eta(t) := \eta_+(t) - \eta_-(t)$, $t \in \mathbb{R}$, we arrive at the representation (5.20). \square

Next we apply Lemma 5.6 to a pair $\{\tilde{A}, H\}$ of extensions of A where \tilde{A} is maximal accumulative and H is selfadjoint.

Theorem 5.7. *Let $\tilde{A}, H \in \text{Ext } A$, $H = H^*$ and let \tilde{A} be a m -accumulative extension. If the condition*

$$(\tilde{A} - \zeta)^{-1} - (H - \zeta)^{-1} \in \mathfrak{S}_1(\mathfrak{H}), \quad \zeta \in \rho(\tilde{A}) \cap \rho(H). \quad (5.27)$$

is satisfied, then the following holds:

(i) *There exists a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ of A^* , which can be chosen regular for $\{\tilde{A}, H\}$, such that $\{\tilde{A}, H\} \in \mathfrak{D}^\Pi$.*

(ii) *If $\{\tilde{A}, H\} \in \mathfrak{D}^\Pi$, then there exists a complex constant c and a complex-valued function $\omega(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ satisfying $\text{Im}(\omega(t)) \leq 0$ for a.e. $t \in \mathbb{R}$ and such that the representation*

$$\Delta_{\tilde{A}/H}^\Pi(z) = c \exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \omega(t) dt \right\}, \quad z \in \mathbb{C}_+, \quad (5.28)$$

is valid.

(iii) *The trace formula*

$$\mathrm{tr} \left((\tilde{A} - z)^{-1} - (H - z)^{-1} \right) = -\frac{1}{\pi} \int_{\mathbb{R}} \frac{\omega(t)}{(t - z)^2} dt, \quad z \in \mathbb{C}_+, \quad (5.29)$$

holds where $\omega(\cdot)$ is given by (ii).

Proof. (i) Since $H = H^*$ is almost solvable and $\rho(\tilde{A}) \cap \rho(H) \supset \mathbb{C}_+$ the existence of a regular boundary triplet Π for A^* is implied by Proposition 4.3 and condition (5.27).

(ii) Let us assume that Π is regular for $\{\tilde{A}, H\}$. If $\tilde{A} = A_B$ and $H = A_C$, where $B, C \in [\mathcal{H}]$, then, by Proposition 2.3, the operator B is accumulative and $C = C^*$. From $B - C \in \mathfrak{S}_1(\mathcal{H})$, cf. Proposition 2.6(ii), we get $B_R - C \in \mathfrak{S}_1(\mathcal{H})$, $B_R := \mathrm{Re}(B)$ and $B_I \in \mathfrak{S}_1(\mathcal{H})$, $B_I := \mathrm{Im}(B) \leq 0$. Consider the selfadjoint extension $\tilde{A}_R := A_{B_R}$. Hence, $\{\tilde{A}, \tilde{A}_R\} \in \mathfrak{D}^\Pi$ and $\{\tilde{A}_R, H\} \in \mathfrak{D}^\Pi$ as well as the following relation

$$\Delta_{\tilde{A}/H}^\Pi(z) = \Delta_{\tilde{A}/\tilde{A}_R}^\Pi(z) \Delta_{\tilde{A}_R/H}^\Pi(z), \quad z \in \mathbb{C}_+. \quad (5.30)$$

holds. Applying Theorem 5.3(ii) we get the existence of a real number c_R and a real-valued function $\xi(\cdot) = \bar{\xi}(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ such that

$$\Delta_{\tilde{A}_R/H}^\Pi(z) = c_R \exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) \xi(t) dt \right\}, \quad z \in \mathbb{C}_+, \quad (5.31)$$

Let us consider $\Delta_{\tilde{A}_R/\tilde{A}}^\Pi(z)$, $z \in \mathbb{C}_+$. We have

$$\Delta_{\tilde{A}_R/\tilde{A}}^\Pi(z) = \det(I - iB_I(B - M(z))^{-1}), \quad z \in \mathbb{C}_+.$$

By Lemma 5.6(i) there exists a complex number $\varkappa \in \mathbb{T}$ and a non-negative function $\eta_+(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ such that the representation

$$\Delta_{\tilde{A}_R/\tilde{A}}^\Pi(z) = \varkappa \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) \eta_+(t) dt \right\}, \quad z \in \mathbb{C}_+, \quad (5.32)$$

holds. Hence,

$$\Delta_{\tilde{A}/\tilde{A}_R}^\Pi(z) = \bar{\varkappa} \exp \left\{ -\frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) \eta_+(t) dt \right\}, \quad z \in \mathbb{C}_+. \quad (5.33)$$

Setting $\omega(t) := \xi(t) - i\eta_+(t)$, $t \in \mathbb{R}$, and inserting (5.31) and (5.33) into (5.30) we arrive at the representation (5.28) provided that Π is regular for $\{H, \tilde{A}\}$. Notice $\mathrm{Im}(\omega(t)) \leq 0$ for a.e. $t \in \mathbb{R}$. Finally, applying Proposition 4.5 we verify the representation (5.28) for any boundary triplet Π such that $\{\tilde{A}, H\} \in \mathfrak{D}^\Pi$. Moreover, the condition $\mathrm{Im}(\omega(t)) \leq 0$ for a.e. $t \in \mathbb{R}$ remains valid.

(iii) The trace formula (5.29) follows immediately from (5.28) and Proposition 4.8(v). \square

In the following a complex-valued function $\omega(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ such that the trace formula (5.29) takes place is called a spectral shift function for the pair $\{\tilde{A}, H\}$ consisting of an accumulative and a selfadjoint extension.

Theorem 5.8. *Let $\tilde{A}_1, \tilde{A}_2 \in \text{Ext}_A$ be m -accumulative and let $H \in \text{Ext}_A$ be a selfadjoint extension such that both pairs $\{\tilde{A}_1, H\}$ and $\{\tilde{A}_2, H\}$ be are resolvent comparable, that is, condition (5.27) is satisfied for both pairs. Then the following holds:*

(i) *If $\lambda_0 \in \rho(\tilde{A}_1) \cap \rho(\tilde{A}_2) \cap \mathbb{R}$ and the inequality*

$$\text{Re}((\tilde{A}_2 - \lambda_0)^{-1}) \leq \text{Re}((\tilde{A}_1 - \lambda_0)^{-1}) \quad (5.34)$$

holds, then there are complex-valued spectral shift functions $\omega_1(\cdot)$ and $\omega_2(\cdot)$ of the pairs $\{\tilde{A}_1, H\}$ and $\{\tilde{A}_2, H\}$, respectively, such that $\text{Re}(\omega_1(t)) \leq \text{Re}(\omega_2(t))$ for a.e. $t \in \mathbb{R}$.

(ii) *If $\lambda_0 \in \rho(\tilde{A}_1) \cap \rho(\tilde{A}_2) \cap \mathbb{R}$ and the inequality*

$$\text{Im}((\tilde{A}_2 - \lambda_0)^{-1}) \leq \text{Im}((\tilde{A}_1 - \lambda_0)^{-1}), \quad (5.35)$$

then there are complex-valued spectral shift functions $\omega_1(\cdot)$ and $\omega_2(\cdot)$ of the pairs $\{\tilde{A}_1, H\}$ and $\{\tilde{A}_2, H\}$, respectively, such that $\text{Im}(\omega_1(t)) \leq \text{Im}(\omega_2(t)) \leq 0$ for a.e. $t \in \mathbb{R}$.

(iii) *If both conditions (5.34) and (5.35) are valid, then there are complex-valued spectral shift functions $\omega_1(\cdot)$ and $\omega_2(\cdot)$ of the pairs $\{\tilde{A}_1, H\}$ and $\{\tilde{A}_2, H\}$, respectively, such that both inequalities $\text{Re}(\omega_1(t)) \leq \text{Re}(\omega_2(t))$ and $\text{Im}(\omega_1(t)) \leq \text{Im}(\omega_2(t)) \leq 0$ are satisfied for a.e. $t \in \mathbb{R}$.*

Proof. (i) By Proposition 3.8, the family $\{\tilde{A}_2, \tilde{A}_1, H\}$ is jointly almost solvable with respect to λ_0 . Taking into account Proposition 3.3 there is a boundary triplet Π for A^* which is regular for $\{\tilde{A}_2, \tilde{A}_1, H\}$ and $\lambda_0 \in \rho(A_0)$. Hence there exist bounded operators $C, B_2, B_1 \in [\mathcal{H}]$ such that $H = A_C$, $\tilde{A}_j = A_{B_j}$, $j = 1, 2$. By Proposition 2.3, $C = C^*$ and the operators B_2, B_1 are accumulative. Moreover, Proposition 2.6(ii) and condition (5.27) yield $B_j - C \in \mathfrak{S}_1(\mathcal{H})$, $j = 1, 2$, and $B_2 - B_1 \in \mathfrak{S}_1(\mathcal{H})$. Hence $\{\tilde{A}_j, H\} \in \mathfrak{D}^\Pi$, $j = 1, 2$, and $\{\tilde{A}_2, \tilde{A}_1\} \in \mathfrak{D}^\Pi$. Notice that $B_{j,I} := \text{Im}(B_j) \in \mathfrak{S}_1(\mathcal{H})$, $j = 1, 2$, $j = 1, 2$.

It follows from the Krein-type formula (2.7) that

$$\begin{aligned} (\tilde{A}_1 - \lambda_0)^{-1} - (\tilde{A}_2 - \lambda_0)^{-1} = \\ \gamma(\lambda_0) \left((B_1 - M(\lambda_0))^{-1} - (B_2 - M(\lambda_0))^{-1} \right) \gamma(\lambda_0)^*. \end{aligned} \quad (5.36)$$

Hence

$$\text{Re}((\tilde{A}_1 - \lambda_0)^{-1}) - \text{Re}((\tilde{A}_2 - \lambda_0)^{-1}) = \gamma(\lambda_0) \left(\text{Re}(\tilde{B}_2) - \text{Re}(\tilde{B}_1) \right) \gamma(\lambda_0)^*$$

where $\tilde{B}_j := -(B_j - M(\lambda_0))^{-1}$, $j = 1, 2$. This identity yields the equivalence

$$\text{Re}((\tilde{A}_2 - \lambda_0)^{-1}) \leq \text{Re}((\tilde{A}_1 - \lambda_0)^{-1}) \iff \text{Re}(\tilde{B}_1) \leq \text{Re}(\tilde{B}_2). \quad (5.37)$$

Notice that the operators \tilde{B}_1 and \tilde{B}_2 are accumulative simultaneously with B_1 and B_2 . Furthermore it holds $\tilde{B}_{j,I} := \text{Im}(\tilde{B}_{j,I}) \in \mathfrak{S}_1(\mathcal{H})$, $j = 1, 2$.

Introducing the modified boundary triplet $\tilde{\Pi} = \{\mathcal{H}, \tilde{\Gamma}_0, \tilde{\Gamma}_1\}$ defined by (5.15) we find that $\tilde{A}_1 = A_{\tilde{B}_1}$ and $\tilde{A}_2 = A_{\tilde{B}_2}$. Next we set $\tilde{A}_3 := A_{\tilde{B}_3}$, where

$$\tilde{B}_3 := \tilde{B}_{2,R} + i\tilde{B}_{1,I}, \quad (5.38)$$

and $\tilde{B}_{j,R} := \text{Re}(\tilde{B}_j)$ and $\tilde{B}_{j,I} := \text{Im}(\tilde{B}_j) \leq 0$, $j = 1, 2$. By Proposition 2.3(iii), $A_{\tilde{B}_3}$ is m -accumulative since $\tilde{B}_{3,I} \leq 0$. One easily verifies that $\{\tilde{A}_2, \tilde{A}_3\} \in \mathfrak{D}^{\tilde{\Pi}}$ and $\{\tilde{A}_3, \tilde{A}_1\} \in \mathfrak{D}^{\tilde{\Pi}}$. Indeed we have $\tilde{B}_2 - \tilde{B}_3 = \tilde{B}_{2,I} - \tilde{B}_{1,I} \in \mathfrak{S}_1(\mathcal{H})$ and $\tilde{B}_3 - \tilde{B}_1 = \tilde{B}_{2,R} - \tilde{B}_{1,R} \in \mathfrak{S}_1(\mathcal{H})$. It follows from the chain rule (4.9) that

$$\Delta_{\tilde{A}_2/H}^{\tilde{\Pi}}(z) = \Delta_{\tilde{A}_2/\tilde{A}_1}^{\tilde{\Pi}}(z) \Delta_{\tilde{A}_1/H}^{\tilde{\Pi}}(z) = \Delta_{\tilde{A}_2/\tilde{A}_3}^{\tilde{\Pi}}(z) \Delta_{\tilde{A}_3/\tilde{A}_1}^{\tilde{\Pi}}(z) \Delta_{\tilde{A}_1/H}^{\tilde{\Pi}}(z), \quad (5.39)$$

$z \in \mathbb{C}_+$.

By Theorem 5.7(ii), the perturbation determinant $\Delta_{\tilde{A}_1/H}^{\tilde{\Pi}}(\cdot)$ admits the representation

$$\Delta_{\tilde{A}_1/H}^{\tilde{\Pi}}(z) = c_1 \exp \left\{ \frac{1}{\pi} \int_R \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \omega_1(t) dt \right\}, \quad z \in \mathbb{C}_+, \quad (5.40)$$

where $c_1 \in \mathbb{C}$, $\omega_1(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ and $\text{Im}(\omega_1(t)) \leq 0$ for a.e. $t \in \mathbb{R}$.

Further, denoting by $\tilde{M}(\cdot)$ the Weyl function corresponding to $\tilde{\Pi}$, we have, by definition,

$$\Delta_{\tilde{A}_3/\tilde{A}_1}^{\tilde{\Pi}}(z) = \det(I + (\tilde{B}_3 - \tilde{B}_1)(\tilde{B}_1 - \tilde{M}(z))^{-1}), \quad z \in \mathbb{C}_+.$$

Since the operators \tilde{A}_1 and \tilde{A}_2 are resolvent comparable, Proposition 2.6(ii) combining with (5.38) yields $\tilde{B}_3 - \tilde{B}_1 = \tilde{B}_{2,R} - \tilde{B}_{1,R} \in \mathfrak{S}_1(\mathcal{H})$. It follows from (5.34), (5.37) and (5.38) that $\tilde{B}_3 - \tilde{B}_1 = \tilde{B}_{2,R} - \tilde{B}_{1,R} \geq 0$. Therefore Lemma 5.2(i) implies the existence of a non-negative real number $\tilde{c}_R > 0$ and a non-negative function $\xi(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ such that the representation

$$\Delta_{\tilde{A}_3/\tilde{A}_1}^{\tilde{\Pi}}(z) = \tilde{c}_R \exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{1}{1+t^2} \right) \xi(t) dt \right\}, \quad z \in \mathbb{C}_+, \quad (5.41)$$

holds. Since

$$\Delta_{\tilde{A}_2/\tilde{A}_3}^{\tilde{\Pi}}(z) = \det(I + (\tilde{B}_2 - \tilde{B}_3)(\tilde{B}_2 - \tilde{M}(z))^{-1}), \quad z \in \mathbb{C}_+,$$

and $\tilde{B}_2 - \tilde{B}_3 = i(B_{2,I} - B_{1,I})$ and $B_{2,I} - B_{1,I} \leq -B_{1,I}$ we get from Lemma 5.6(ii) the existence of a unimodular constant $\tilde{\varkappa} \in \mathbb{T}$ and a real-valued function $\eta(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ such that the representation

$$\Delta_{\tilde{A}_2/\tilde{A}_3}^{\tilde{\Pi}}(z) = \tilde{\varkappa} \exp \left\{ \frac{i}{\pi} \int_R \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \eta(t) dt \right\}, \quad z \in \mathbb{C}_+, \quad (5.42)$$

is valid. Inserting (5.40), (5.41) and (5.42) into (5.39) and setting $c_2 := c_1 \tilde{c}_R \tilde{\mathcal{K}} \in \mathbb{C}$ and $\omega_2(t) := \omega_1(t) + \xi_+(t) + i\eta(t)$, $t \in \mathbb{R}$, we arrive at the representation

$$\Delta_{\tilde{A}_2/H}^{\tilde{\Pi}}(z) = c_2 \exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \omega_2(t) dt \right\}, \quad z \in \mathbb{C}_+, \quad (5.43)$$

where $\operatorname{Re}(\omega_1(t)) \leq \operatorname{Re}(\omega_1(t) + \xi_+(t)) = \operatorname{Re}(\omega_2(t))$ for a.e. $t \in \mathbb{R}$.

(ii) To handle the second case we use again the factorization (5.39) but in slightly different manner. From Theorem 5.7(ii) it follows the representation (5.43) where $c_2 \in \mathbb{C}$, $\omega_2(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ and $\operatorname{Im}(\omega_2(t)) \leq 0$ for a.e. $t \in \mathbb{R}$. Now representation (5.41) follows from Lemma 5.2(ii) where, however, the function $\xi(\cdot)$ is not necessarily non-negative. It follows from (5.36) and the assumption (5.35) that

$$\operatorname{Im}((\tilde{A}_2 - \lambda_0)^{-1}) \leq \operatorname{Im}((\tilde{A}_1 - \lambda_0)^{-1}) \iff \operatorname{Im}(\tilde{B}_1) \leq \operatorname{Im}(\tilde{B}_2)$$

which yields $\tilde{B}_{2,I} - \tilde{B}_{1,I} \geq 0$. By Lemma 5.6(i), representation (5.42) holds with a non-negative function $\eta(\cdot) \geq 0$. Inserting (5.43), (5.41) and (5.42) into (5.39) and setting $\omega_1(t) := \omega_2(t) - \xi(t) - i\eta(t)$, $t \in \mathbb{R}$, we obtain the representation (5.40). From (5.40) and Proposition 4.8(v) we obtain that $\omega_1(t)$ is a spectral shift function for the pair $\{\tilde{A}_1, H\}$. Obviously, we have $\operatorname{Im}(\omega_1(t)) \leq \operatorname{Im}(\omega_1(t) + \eta(t)) = \operatorname{Im}(\omega_2(t)) \leq 0$ for a.e. $t \in \mathbb{R}$.

(iii) This statement can be proved following the reasoning of (ii). Since in addition the condition (5.34) is satisfied we find that the representation (5.41) holds and $\xi(t) \geq 0$ for a.e. $t \in \mathbb{R}$. Since $\omega_1(t) = \omega_2(t) - \xi(t) - i\eta(t)$, $t \in \mathbb{R}$, we easily verify $\operatorname{Re}(\omega_1(t)) \leq \operatorname{Re}(\omega_2(t))$ and $\operatorname{Im}(\omega_1(t)) \leq \operatorname{Im}(\omega_2(t))$ for a.e. $t \in \mathbb{R}$. \square

Remark 5.9.

(i) By Lemma 5.1 the trace formula (5.9) for selfadjoint extensions determines the function $\xi(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ uniquely up to a real constant. In contrast to that, the trace formula (5.29) does not determine the spectral shift function $\omega(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ up to a constant.

(ii) Trace formula (5.29) differs from that one of (5.9) because the spectral shift function $\omega(\cdot)$ is not real-valued. Using the results of [2] one gets a trace formula of type (5.9) for the pair $\{\tilde{A}, H\}$ if in addition to the assumption (5.27) the condition

$$(\tilde{A}^* + i)^{-1} - (\tilde{A} - i)^{-1} + 2i(\tilde{A}^* + i)^{-1}(\tilde{A} - i)^{-1} \in \mathfrak{S}_1^0(\mathfrak{H}) \quad (5.44)$$

is satisfied. Here $\mathfrak{S}_1^0(\mathfrak{H})$ stands for the ideal of all compact operators T satisfying

$$\sum_{k=1}^{\infty} s_k(T) \log^+ \left(\frac{1}{s_k(T)} \right) < \infty,$$

where $s_k(T)$, $k \in \mathbb{N}$, are the singular numbers of T . Note that $\mathfrak{S}_1^0(\mathfrak{H})$ is a strict part of $\mathfrak{S}_1(\mathfrak{H})$. In this case there exists a *real-valued* function $\vartheta(\cdot) \in$

$L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ such that the trace formula

$$\mathrm{tr} \left((\tilde{A} - z)^{-1} - (H - z)^{-1} \right) = -\frac{1}{\pi} \int_{\mathbb{R}} \frac{\vartheta(t)}{(t - z)^2} dt, \quad z \in \mathbb{C}_+,$$

holds. If $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^* , which is regular for $\{\tilde{A}, H\}$. If $\tilde{A} = A_B$ and $H = A_C$, $B, C \in \mathcal{H}$, then condition (5.44) is equivalent to $B_I = \mathrm{Im}(B) \in \mathfrak{S}_1^0(\mathcal{H})$. Assumption (5.27) implies $B_R - C \in \mathfrak{S}_1(\mathcal{H})$.

(iii) For further results on trace formulas for non-selfadjoint operators we refer to papers of A. Rybkin [56, 55, 57, 58, 59].

Let us extend the results to maximal accumulative extensions.

Theorem 5.10. *Let A be as above and let $\tilde{A}_1, \tilde{A}_2 \in \mathrm{Ext}_A$ be maximal accumulative extensions of A such that*

$$(\tilde{A}_2 - \zeta)^{-1} - (\tilde{A}_1 - \zeta)^{-1} \in \mathfrak{S}_1(\mathfrak{H}), \quad \zeta \in \rho(\tilde{A}_2) \cap \rho(\tilde{A}_1). \quad (5.45)$$

and $\rho(\tilde{A}_1) \cap \mathbb{C}_- \neq \emptyset$. Then the following assertions are valid:

(i) *There exists a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* , which can be chosen regular for $\{\tilde{A}_2, \tilde{A}_1\}$, such that $\{\tilde{A}_2, \tilde{A}_1\} \in \mathfrak{D}^\Pi$.*

(ii) *If $\{\tilde{A}_2, \tilde{A}_1\} \in \mathfrak{D}^\Pi$, then there exist a complex-valued function $\omega(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ and a constant $c \in \mathbb{C}$ such that the representation*

$$\Delta_{\tilde{A}_2/\tilde{A}_1}^\Pi(z) = c \exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) \omega(t) dt \right\}, \quad z \in \mathbb{C}_+, \quad (5.46)$$

holds.

(iii) *The trace formula*

$$\mathrm{tr} \left((\tilde{A}_2 - z)^{-1} - (\tilde{A}_1 - z)^{-1} \right) = -\frac{1}{\pi} \int_{\mathbb{R}} \frac{\omega(t)}{(t - z)^2} dt, \quad z \in \mathbb{C}_+. \quad (5.47)$$

holds where $\omega(\cdot)$ is given by (ii).

Proof. (i) Since $\rho(\tilde{A}_1) \cap \rho(\tilde{A}_2) \supset \mathbb{C}_+$ there exists a boundary triplet Π by Corollary 4.4(i) which is regular for the pair $\{\tilde{A}_2, \tilde{A}_1\}$. Hence there exist accumulative operators $B_j \in [\mathcal{H}]$ such that $\tilde{A}_j = A_{B_j}$, $j = 1, 2$. By Proposition 2.6(ii) condition (5.45) is equivalent to $B_2 - B_1 \in \mathfrak{S}_1(\mathcal{H})$ which yields the inclusion $\{\tilde{A}_2, \tilde{A}_1\} \in \mathfrak{D}^\Pi$.

(ii) First, let us assume that $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is regular for $\{\tilde{A}_2, \tilde{A}_1\}$. Clearly, we have $B_{2,R} - B_{1,R} \in \mathfrak{S}_1(\mathcal{H})$ and $B_{2,I} - B_{1,I} \in \mathfrak{S}_1(\mathcal{H})$. We set

$$B_3 = B_{2,R} + iB_{1,I} \quad (5.48)$$

and $\tilde{A}_3 := A_{B_3}$, $\mathrm{dom}(A_3) = \ker(\Gamma_1 - B_3\Gamma_0)$. Since B_3 is accumulative, \tilde{A}_3 is m -accumulative (cf. Proposition 2.3). Since $B_2 - B_3 = i(B_{2,I} - B_{1,I}) \in$

$\mathfrak{S}_1(\mathcal{H})$, $\{\tilde{A}_2, \tilde{A}_3\} \in \mathfrak{D}^\Pi$. Furthermore, $B_3 - B_1 = B_{2,R} - B_{1,R} \in \mathfrak{S}_1(\mathcal{H})$, hence $\{\tilde{A}_3, \tilde{A}_1\} \in \mathfrak{D}^\Pi$. Therefore the perturbation determinant $\Delta_{\tilde{A}_2/\tilde{A}_3}^\Pi(\cdot)$ is well defined,

$$\begin{aligned}\Delta_{\tilde{A}_2/\tilde{A}_3}^\Pi(z) &= \det(I + (B_2 - B_3)(B_3 - M(z))^{-1}) \\ &= \det(I + i(B_{2,I} - B_{1,I})(B_3 - M(z))^{-1}), \quad z \in \mathbb{C}_+.\end{aligned}$$

Since $B_{2,I} - B_{1,I} \leq -B_{1,I} = -B_{3,I}$, we obtain from Lemma 5.6(ii) that there exist a complex number $\varkappa \in \mathbb{T}$ and a *real-valued* function $\eta(\cdot) \in L^2(\mathbb{R}, \frac{1}{1+t^2}dt)$ such that the representation

$$\Delta_{\tilde{A}_2/\tilde{A}_3}^\Pi(z) = \varkappa \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \eta(t) dt \right\}, \quad z \in \mathbb{C}_+, \quad (5.49)$$

holds. Further, it follows from (5.48) that

$$\begin{aligned}\Delta_{\tilde{A}_3/\tilde{A}_1}^\Pi(z) &= \det(I + (B_3 - B_1)(B_1 - M(z))^{-1}) \\ &= \det(I + (B_{2,R} - B_{1,R})(B_1 - M(z))^{-1}), \quad z \in \mathbb{C}_+.\end{aligned}$$

By Lemma 5.2(ii), there exist a constant $c_1 > 0$ and a *real-valued* function $\xi(\cdot) \in L^2(\mathbb{R}, \frac{1}{1+t^2}dt)$ such that the representation

$$\Delta_{\tilde{A}_3/\tilde{A}_1}^\Pi(z) = c_1 \exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \xi(t) dt \right\}, \quad z \in \mathbb{C}_+, \quad (5.50)$$

holds. Combining (5.49) with (5.50), applying the identity $\Delta_{\tilde{A}_2/\tilde{A}_3}^\Pi(z) \Delta_{\tilde{A}_3/\tilde{A}_1}^\Pi(z) = \Delta_{\tilde{A}_2/\tilde{A}_1}^\Pi(z)$, $z \in \mathbb{C}_+$, and setting $c_0 := c_1 \varkappa$ and $\omega(t) := \xi(t) + i\eta(t)$, $t \in \mathbb{R}$, we arrive at the representation (5.46) with c_0 in place of c . Finally, if Π is not regular we apply Proposition 4.5 to get (5.46).

(iii) The trace formula (5.47) follows immediately from (5.46) and Proposition 4.8(v). \square

Corollary 5.11. *Let $\tilde{A}_2, \tilde{A}_1 \in \text{Ext}_A$ be accumulative extensions of A such that the pair $\{\tilde{A}_2, \tilde{A}_1\}$ is resolvent comparable, that is condition (5.45) is satisfied.*

- (i) *If (5.34) is satisfied, then there is a complex-valued spectral shift function $\omega(\cdot)$ of the pair $\{\tilde{A}_2, \tilde{A}_1\}$ such that $\text{Re}(\omega(t)) \geq 0$ for a.e. $t \in \mathbb{R}$.*
- (ii) *If (5.35) is satisfied, then there is a complex-valued spectral shift function $\omega(\cdot)$ of the pair $\{\tilde{A}_2, \tilde{A}_1\}$ such that $\text{Im}(\omega(t)) \geq 0$ for a.e. $t \in \mathbb{R}$.*
- (iii) *If (5.34) and (5.35) are satisfied, then there is a complex-valued spectral shift function $\omega(\cdot)$ of the pair $\{\tilde{A}_2, \tilde{A}_1\}$ such that $\text{Re}(\omega(t)) \geq 0$ and $\text{Im}(\omega(t)) \geq 0$ for a.e. $t \in \mathbb{R}$.*

Proof. By Corollary 4.4(ii) there is a boundary triplet Π for A^* which is regular for $\{\tilde{A}_2, \tilde{A}_1\}$ such that $\lambda \in \rho(A_0)$ if where $\lambda \in \rho(\tilde{A}_2) \cap \rho(\tilde{A}_1) \cap \mathbb{R}$. Hence there are bounded accumulative operators B_2 and B_1 such that $\tilde{A}_j = A_{B_j}$, $j = 1, 2$.

Let us introduce the boundary triplet (5.15). Setting $\tilde{B}_j = -(B_j - M(\lambda_0))^{-1}$, $j = 1, 2$, we have $\tilde{A}_j = A_{\tilde{B}_j}$, $j = 1, 2$. In addition we introduce \tilde{B}_3 defined by (5.38) and the maximal accumulative extension $\tilde{A}_3 := A_{\tilde{B}_3}$. Now we follow the proof of Theorem 5.8.

(i) In this case we get $\tilde{B}_{1,R} \leq \tilde{B}_{2,R}$. As above, this yields that $\xi(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ can be chosen non-negative in the representation (5.41). Using $\Delta_{\tilde{A}_2/\tilde{A}_1}^{\tilde{\Pi}}(z) = \Delta_{\tilde{A}_2/\tilde{A}_3}^{\tilde{\Pi}}(z) \Delta_{\tilde{A}_3/\tilde{A}_1}^{\tilde{\Pi}}(z)$, $z \in \mathbb{C}_+$, taking into account the representations (5.41) and (5.42) and setting $\omega(t) := \xi(t) + i\eta(t)$, $t \in \mathbb{R}$, we get

$$\Delta_{\tilde{A}_2/\tilde{A}_1}^{\tilde{\Pi}}(z) = \tilde{c} \exp\left\{\frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{1}{1+t^2}\right) \omega(t) dt, \quad z \in \mathbb{C}_+, \quad (5.51)\right.$$

$\tilde{c} \in \mathbb{C}$ where $\operatorname{Re}(\omega(t)) \geq 0$ for a.e. $t \in \mathbb{R}$. Hence there is a spectral shift function satisfying $\operatorname{Re}(\omega(t)) \geq 0$ for a.e. $t \in \mathbb{R}$.

(ii) In this case we have $\tilde{B}_{2,I} \leq \tilde{B}_{1,I}$. This yields that $\eta(\cdot)$ in the representation (5.42) can be chosen non-negative. This immediately yields that $\operatorname{Im}(\omega(t)) \geq 0$ for a.e. $t \in \mathbb{R}$ in the representation (5.51). Hence there is a spectral shift function satisfying $\operatorname{Im}(\omega(t)) \geq 0$ for a.e. $t \in \mathbb{R}$.

(iii) Finally, in this case $\xi(\cdot) \geq 0$ and $\eta(\cdot)$ can be chosen non-negative in the representation (5.41) and (5.42), respectively. Setting $\omega(t) := \xi(t) + i\eta(t)$, $t \in \mathbb{R}$, we verify (5.51). Hence there is a spectral shift function satisfying $\operatorname{Re}(\omega(t)) \geq 0$ and $\operatorname{Im}(\omega(t)) \geq 0$ for a.e. $t \in \mathbb{R}$. \square

5.3 Pairs of extensions with one m -accumulative operator

Here we consider trace formulas for pairs $\{\tilde{A}', \tilde{A}\}$ of proper extensions of a closed symmetric operator A assuming that \tilde{A}' is m -accumulative extension.

Lemma 5.12. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* , $M(\cdot)$ the corresponding Weyl function and let $B \in [\mathcal{H}]$ be an accumulative operator, i.e. $B_I \leq 0$.*

(i) *If $0 \leq V_+ \leq 2|B_I| = -2B_I$, $V \in \mathfrak{S}_1(\mathcal{H})$, then the holomorphic function $w_+(z) := \det(I + iV_+(B - M(z))^{-1})$, $z \in \mathbb{C}_+$, is contractive. In particular, there exist a non-negative Borel measure $\mu_+(\cdot)$ satisfying $\int_{\mathbb{R}} \frac{1}{1+t^2} d\mu_+(t) < \infty$ as well as numbers $\varkappa_+ \in \mathbb{T}$ and $\alpha_+ \geq 0$ such that the following representation*

$$w_+(z) = \varkappa_+ \mathcal{B}_+(z) \exp\left\{\frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu_+(t)\right\} e^{i\alpha_+ z} \quad (5.52)$$

$z \in \mathbb{C}_+$, holds where $\mathcal{B}_+(\cdot)$ is the Blaschke product formed by the zeros $\{z_k^+\}_{k \in \mathbb{N}}$ of $w_+(\cdot)$ in \mathbb{C}_+ , cf. (D.2).

(ii) *If $V \leq 2|B_I| = -2B_I$ and $V \in \mathfrak{S}_1(\mathcal{H})$, then the perturbation determinant $w(\cdot) := \det(I + iV(B - M(\cdot))^{-1})$ belongs to the Smirnov class $\mathcal{N}^+(\mathbb{C}_+)$, see Appendix D. In particular, there exist a non-negative Borel measure $\mu_+(\cdot) \geq 0$*

satisfying $\int_{\mathbb{R}} \frac{1}{1+t^2} d\mu_+(t) < \infty$, a non-negative function $\eta(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ as well as numbers $\varkappa \in \mathbb{T}$, $\alpha \geq 0$ such that

$$w(z) = \varkappa \mathcal{B}_+(z) \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu(t) \right\} e^{i\alpha z}, \quad z \in \mathbb{C}_+, \quad (5.53)$$

where $\mu(\cdot) := \mu_+(\cdot) - \eta(\cdot)dt$ and $\mathcal{B}_+(z)$ is the Blaschke product formed by the zeros of $w(\cdot)$ lying in \mathbb{C}_+ .

Proof. (i) We introduce the holomorphic operator-valued function

$$W_+(z) := I + i\sqrt{V_+}(B - M(z))^{-1}\sqrt{V_+}, \quad z \in \mathbb{C}_+. \quad (5.54)$$

Since $B_I \leq 0$, $\text{Im}(M(z)) > 0$ and $0 \in \rho(\text{Im}(M(z)))$ for $z \in \mathbb{C}_+$, the operator $(B - M(z))^{-1}$ is well-defined and bounded for $z \in \mathbb{C}_+$. Further, we have

$$\begin{aligned} I - W_+(z)^*W_+(z) &= i\sqrt{V_+}((B^* - M(z)^*)^{-1} - (B - M(z))^{-1})\sqrt{V_+} \\ &\quad - \sqrt{V_+}(B^* - M(z)^*)^{-1}V_+(B - M(z))^{-1}\sqrt{V_+}. \end{aligned} \quad (5.55)$$

Noting that

$$\begin{aligned} (B^* - M(z)^*)^{-1} - (B - M(z))^{-1} &= -2i(B^* - M(z)^*)^{-1} \cdot (|B_I| + M_I(z)) \cdot (B - M(z))^{-1}, \end{aligned} \quad (5.56)$$

we obtain from (5.55) that

$$\begin{aligned} I - W_+(z)^*W_+(z) &= \sqrt{V_+}(B^* - M^*(z))^{-1} \cdot (2|B_I| - V_+ + 2\text{Im}(M(z))) \cdot (B - M(z))^{-1}\sqrt{V_+}. \end{aligned}$$

Since $V_+ \leq 2|B_I|$ and $\text{Im}(M(z))$ is positively definite, we have $I - W_+(z)^*W_+(z) \geq 0$ for $z \in \mathbb{C}_+$, i.e. $W_+(\cdot)$ is contractive in \mathbb{C}_+ . Hence $w_+(\cdot) = \det W_+(\cdot)$ is contractive in \mathbb{C}_+ . Now the representation (5.52) immediately follows from the factorization (D.3).

(ii) Let $V = V_+ - V_-$, $V_{\pm} \geq 0$. We set $B_- := B - iV_-$. Since $(B_-)_I = B_I - V_- \leq 0$, the operator B_- is accumulative too. According to (5.6) we get

$$\det(I + iV(B - M(z))^{-1}) = \frac{\det(I + iV_+(B_- - M(z))^{-1})}{\det(I + iV_-(B_- - M(z))^{-1})}, \quad z \in \mathbb{C}_+. \quad (5.57)$$

The assumption $V \leq -2B_I$ yields $0 \leq V_+ \leq -2B_I + V_- \leq -2B_I + 2V_- = -2(B_-)_I$. Applying statement (i) to the operators B_- and V_+ we obtain that $\det(I + iV_+(B_- - M(z))^{-1})$ is a contractive analytic function. Furthermore, from $0 \leq V_- \leq -B_I + V_- = -(B_-)_I$ and Lemma 5.6(i) we get that $\det(I + iV_-(B_- - M(z))^{-1})$ is an outer function. Applying Lemma D.1 we complete the proof. \square

Theorem 5.13. Let $\tilde{A}', \tilde{A} \in \text{Ext}_A$ and let \tilde{A} be an m -accumulative extension with $\rho(\tilde{A}) \cap \mathbb{C}_- \neq \emptyset$. Assume in addition, that condition (1.6) is satisfied for some $\zeta \in \rho(\tilde{A}') \cap \rho(\tilde{A})$. Then the following holds:

- (i) There exists a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* , which is regular for $\{\tilde{A}', \tilde{A}\}$ and such that $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$.
- (ii) If $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$, then there exist a non-negative Borel measure $\mu_+(\cdot)$ satisfying $\int \frac{1}{1+t^2} d\mu_+(t) < \infty$ and a complex-valued function $\omega(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ as well as constants $\alpha_+ \geq 0$ and $c \in \mathbb{C}$ such that the representation

$$\Delta_{\tilde{A}'/\tilde{A}}^\Pi(z) = c \mathcal{B}_+(z) \exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\nu(t) \right\} e^{i\alpha_+ z}, \quad (5.58)$$

$z \in \mathbb{C}_+$, holds where $d\nu(\cdot) := \omega(\cdot) dt + i d\mu_+(\cdot)$ and $\mathcal{B}_+(\cdot)$ is the Blaschke product (cf. (D.2)) formed by the eigenvalues z_k^+ of \tilde{A}' in \mathbb{C}_+ and their algebraic multiplicities m_k^+ satisfying condition (D.1).

- (iii) The following trace formula holds

$$\begin{aligned} & \text{tr} \left((\tilde{A}' - z)^{-1} - (\tilde{A} - z)^{-1} \right) \\ &= -2i \sum_k \frac{m_k^+ \text{Im}(z_k^+)}{(z - \bar{z}_k^+)(z - z_k^+)} - \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{(t-z)^2} d\nu(t) - i\alpha_+, \quad z \in \rho(\tilde{A}') \cap \mathbb{C}_+. \end{aligned} \quad (5.59)$$

Proof. (i) Since \tilde{A} is m -accumulative, $\mathbb{C}_+ \subset \rho(\tilde{A})$. Therefore and due to the assumption $\rho(\tilde{A}) \cap \mathbb{C}_- \neq \emptyset$, the conditions of Proposition 3.5 are satisfied, hence the extension \tilde{A} is almost solvable. Now the existence of a regular boundary triplet Π is implied by Corollary 4.4(i) and the assumption (1.6).

(ii) Let Π be regular for $\{\tilde{A}', \tilde{A}\}$. By definition, there exist bounded operators $B', B \in [\mathcal{H}]$, such that $\tilde{A}' = A_{B'}$ and $\tilde{A} = A_B$. Since \tilde{A} is m -accumulative, by Proposition 2.3(iii), the operator B is accumulative too, i.e. $B_I = \text{Im}(B) \leq 0$. By Proposition 2.6(ii) condition (1.6) is equivalent to $B' - B \in \mathfrak{S}_1(\mathcal{H})$. Hence $B'_R - B_R \in \mathfrak{S}_1(\mathcal{H})$ and $V := B'_I - B_I \in \mathfrak{S}_1(\mathcal{H})$. We set

$$C := B'_R + iB_I. \quad (5.60)$$

Since B is accumulative, the operator C is also accumulative, $C_I = B_I \leq 0$. Let $V = B'_I - B_I = V_+ - V_-$, $V_\pm \geq 0$, be the orthogonal decomposition of the operator $V = V^*$. We set

$$D := C - i(V_+ + V_-) \quad (5.61)$$

and note that D is accumulative because so is C and $V_\pm \geq 0$. Since

$$B' - D = B'_R + iB'_I - B'_R - iB_I + i(V_+ + V_-) = 2iV_+ \in \mathfrak{S}_1(\mathcal{H}), \quad (5.62)$$

we get $\{\tilde{A}', A_D\} \in \mathfrak{D}^\Pi$. Notice that

$$2V_+ \leq -2B_I + 2V_+ + 2V_- = -2(B_I - V_+ - V_-) = -2D_I.$$

According to (5.62) one has $\Delta_{\tilde{A}'/A_D}^\Pi(\cdot) = \det(I + 2iV_+(D - M(\cdot))^{-1})$. By Lemma 5.12(i), $\Delta_{\tilde{A}'/A_D}^\Pi(\cdot)$ is contractive in \mathbb{C}_+ and admits the representation

$$\Delta_{\tilde{A}'/A_D}^\Pi(z) = \varkappa_+ \mathcal{B}_+(z) \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu_+(t) \right\} e^{i\alpha_+ z} \quad (5.63)$$

where $\varkappa_+ \in \mathbb{T}$, $\alpha_+ \geq 0$, μ_+ is a non-negative Borel measure on \mathbb{R} satisfying $\int_{\mathbb{R}} \frac{1}{1+t^2} d\mu_+(t) < \infty$ and $\mathcal{B}_+(\cdot)$ is the Blaschke product, cf. (D.2) formed by zeros $z_k^+ \in \mathbb{C}_+$ of $\Delta_{\tilde{A}'/A_D}^\Pi(\cdot)$. By Proposition 4.8(iv), (cf. formula (4.12)), each zero z_k^+ of $\Delta_{\tilde{A}'/A_D}^\Pi(\cdot)$ of the multiplicity m_k^+ is just the eigenvalue of \tilde{A}' lying in \mathbb{C}_+ , and m_k^+ is its algebraic multiplicity. In particular, this yields that $\{z_k^+\}_{k \in \mathbb{N}} = \sigma_p(\tilde{A}') \cap \mathbb{C}_+$, and the eigenvalues $\{z_k^+\}_{k \in \mathbb{N}}$ satisfy condition (D.1).

Furthermore, since $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$ and $\{\tilde{A}', A_D\} \in \mathfrak{D}^\Pi$, we have $\{A_D, \tilde{A}\} \in \mathfrak{D}^\Pi$. Note that the extension A_D is m -accumulative since D is m -accumulative. Thus, both A_D and \tilde{A} are m -accumulative and, by Theorem 5.10(ii), there exists a complex-valued function $\omega(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ and a complex constant $c_D \in \mathbb{C}$ such that the following representation holds

$$\Delta_{A_D/\tilde{A}}^\Pi(z) = c_D \exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \omega(t) dt \right\}, \quad z \in \mathbb{C}_+. \quad (5.64)$$

Using the chain rule $\Delta_{\tilde{A}'/\tilde{A}}^\Pi(z) = \Delta_{\tilde{A}'/A_D}^\Pi(z) \Delta_{A_D/\tilde{A}}^\Pi(z)$, $z \in \rho(\tilde{A}) \cap \mathbb{C}_+$ (cf. Proposition 4.8(ii)), and combining (5.63) with (5.64) we arrive at representation (5.58) with $c := c_D \varkappa_+$ and $d\nu(t) = \omega(t) dt + i d\mu_+(t)$.

The case of a boundary triplet Π which is not regular for the pair $\{\tilde{A}', \tilde{A}\}$ is reduced to the previous one by applying Proposition 4.5.

(iii) Trace formula (5.59) is implied now by combining (5.58) with Proposition 4.8(v). \square

Using the Riesz-Dunford functional calculus, cf. Appendix F, we extend trace formula (5.59) to the case of analytic functions of the class $\mathcal{F}(\tilde{A}, \tilde{A}')$.

Corollary 5.14. *Let the assumptions of Theorem 5.13 be satisfied. Let $\{z_k^+\}_{k \in \mathbb{N}} = \sigma_p(\tilde{A}') \cap \mathbb{C}_+$ and let m_k^+ be the algebraic multiplicity of z_k^+ , $k \in \mathbb{N}$. If $\Phi \in \mathcal{F}(\tilde{A}, \tilde{A}')$, cf. Appendix F, then $\Phi(\tilde{A}') - \Phi(\tilde{A}) \in \mathfrak{S}_1(\mathfrak{H})$ and*

$$\begin{aligned} \text{tr}(\Phi(\tilde{A}') - \Phi(\tilde{A})) = & \quad (5.65) \\ & \sum_k m_k^+ (\Phi(z_k^+) - \Phi(\bar{z}_k^+)) + \frac{1}{\pi} \int_{\mathbb{R}} \Phi'(t) d\nu(t) + i\alpha_+ \text{res}_\infty(\Phi), \end{aligned}$$

where z_k^+ are the eigenvalues of \tilde{A}' in \mathbb{C}_+ and m_k^+ their algebraic multiplicities.

Proof. From Lemma F.1 it follows that $\Phi(\tilde{A}') - \Phi(\tilde{A}) \in \mathfrak{S}_1(\mathfrak{H})$. Multiplying (5.59) with $\Phi(z)$ and integrating both sides with respect to dz we obtain immediately (5.65) using formulas (F.1), (F.2), (F.4) and (F.5) of Appendix F. \square

5.4 Pairs of an extension and its adjoint

Next we consider perturbation determinants and trace formulas for pairs $\{\tilde{A}, \tilde{A}^*\}$ of proper extensions $\tilde{A}, \tilde{A}^* \in \text{Ext}_A$ assuming that $\rho(\tilde{A}) \cap \rho(\tilde{A}^*) \neq \emptyset$.

Theorem 5.15. *Let $\tilde{A} \in \text{Ext}_A$ and $\rho(\tilde{A}) \cap \rho(\tilde{A}^*) \neq \emptyset$. Assume also that*

$$(\tilde{A} - \zeta)^{-1} - (\tilde{A}^* - \zeta)^{-1} \in \mathfrak{S}_1(\mathfrak{H}), \quad \zeta \in \rho(\tilde{A}) \cap \rho(\tilde{A}^*). \quad (5.66)$$

Then the following holds:

- (i) *There exists a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* , which can be chosen regular for $\{\tilde{A}, \tilde{A}^*\}$, such that $\{\tilde{A}, \tilde{A}^*\} \in \mathfrak{D}^\Pi$.*
- (ii) *If $\{\tilde{A}, \tilde{A}^*\} \in \mathfrak{D}^\Pi$ and the triplet Π is regular for the pair $\{\tilde{A}, \tilde{A}^*\}$, then*

$$\Delta_{\tilde{A}/\tilde{A}^*}^\Pi(z) = \det(W_{\tilde{A}}^\Pi(z)), \quad z \in \rho(\tilde{A}^*) \cap \mathbb{C}_\pm, \quad (5.67)$$

where $W_{\tilde{A}}^\Pi(\cdot)$ is the characteristic operator-valued function of \tilde{A} defined by (3.3), cf. Proposition 3.9.

- (iii) *If $\{\tilde{A}, \tilde{A}^*\} \in \mathfrak{D}^\Pi$, then there exist a real-valued measure μ on \mathbb{R} satisfying $\int_{\mathbb{R}} \frac{1}{1+t^2} |d\mu|(t) < \infty$ and constants $\alpha \in \mathbb{R}$ and $c \in \mathbb{C}$ such that the representation*

$$\Delta_{\tilde{A}/\tilde{A}^*}^\Pi(z) = c \frac{\mathcal{B}_+(z)}{\mathcal{B}_-(z)} \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu(t) \right\} e^{i\alpha z}, \quad (5.68)$$

holds for $z \in \rho(\tilde{A}^) \cap \mathbb{C}_+$ where $\mathcal{B}_+(\cdot)$ and $\mathcal{B}_-(\cdot)$ are Blaschke products (cf. (D.2)) formed by the zeros $\{z_k^+\}_{k \in \mathbb{N}}$ and $\{\bar{z}_l^-\}_{l \in \mathbb{N}}$, where $\{z_k^+\}_{k \in \mathbb{N}}$ and $\{\bar{z}_l^-\}_{l \in \mathbb{N}}$ are eigenvalues of \tilde{A} in \mathbb{C}_+ and \mathbb{C}_- , respectively, and m_k^+ and m_l^- their algebraic multiplicities, respectively. Both sequences $\{z_k^+\}_{k \in \mathbb{N}}$ and $\{\bar{z}_l^-\}_{l \in \mathbb{N}}$ satisfy condition (D.1).*

- (iv) *The following trace formula holds*

$$\begin{aligned} & \text{tr} \left((\tilde{A}^* - z)^{-1} - (\tilde{A} - z)^{-1} \right) \\ &= 2i \sum_n \frac{m_n^+ \text{Im}(z_n)}{(z - z_n)(z - \bar{z}_n)} + \frac{i}{\pi} \int_{\mathbb{R}} \frac{1}{(t - z)^2} d\mu(t) + i\alpha, \end{aligned} \quad (5.69)$$

$z \in \rho(\tilde{A}^) \cap \rho(\tilde{A}) \cap \mathbb{C}_+$, where z_n and m_n denote the eigenvalues of \tilde{A} in $\mathbb{C} \setminus \mathbb{R}$ and their algebraic multiplicities, respectively.*

Proof. (i) Let $z_0 \in \rho(\tilde{A}) \cap \rho(\tilde{A}^*)$. Then $\bar{z}_0 \in \rho(\tilde{A}) \cap \rho(\tilde{A}^*)$ and, by Proposition 3.5, the extension \tilde{A} is almost solvable. By Corollary 4.4(i), there exists a boundary triplet Π regular for $\{\tilde{A}, \tilde{A}^*\}$ and satisfying $\{\tilde{A}, \tilde{A}^*\} \in \mathfrak{D}^\Pi$.

(ii) Assume that $\{\tilde{A}, \tilde{A}^*\} \in \mathfrak{D}^\Pi$ and Π is a regular boundary triplet for $\{\tilde{A}, \tilde{A}^*\}$. By definition, there exists a bounded operator $B \in [\mathcal{H}]$ such that $\tilde{A} = A_B$ and $\tilde{A}^* = A_{B^*}$. Let $B_I := J|B_I|$ be the polar decomposition of

$B_I = (B - B^*)/2i = B_I^*$ where $J = J^* = J^{-1}$. From $\{\tilde{A}, \tilde{A}^*\} \in \mathfrak{D}^\Pi$ and Proposition 2.6(ii) we get $B - B^* = 2iB_I \in \mathfrak{S}_1(\mathcal{H})$, $B_I := \text{Im}(B)$. Taking into account the property (A.1) we get

$$\begin{aligned} \Delta_{\tilde{A}/\tilde{A}^*}^\Pi(z) &= \det(I + (B - B^*)(B^* - M(z))^{-1}) \\ &= \det(I + 2i\sqrt{|B_I|}(B^* - M(z))^{-1}\sqrt{|B_I|}J), \quad z \in \rho(\tilde{A}^*) \cap \mathbb{C}_\pm. \end{aligned}$$

Applying Proposition 3.9 we arrive at (5.67).

(iii) Let a boundary triplet Π be regular for $\{\tilde{A}, \tilde{A}^*\}$. Consider the spectral decomposition $B_I = B_I^+ - B_I^-$ of B_I , where B_I^\pm are orthogonal, $B_I^\pm \geq 0$, and $B_I^\pm \in \mathfrak{S}_1(\mathcal{H})$. Alongside B consider the dissipative operator $B_1 = B_R + i|B_I| = B_R + iB_I^+ + iB_I^-$ and the corresponding extension $\tilde{A}' := A_{B_1^*}$. By Proposition 2.3(iii), \tilde{A}' is maximal accumulative. We note that $\{\tilde{A}, \tilde{A}'\} \in \mathfrak{D}^\Pi$. To the perturbation determinant $\Delta_{\tilde{A}/\tilde{A}'}^\Pi(\cdot)$ we can apply Lemma 5.12(i) with B_1^* in place of B and $V_+ := 2B_I^+ \leq -2\text{Im}(B_1^*)$. This yields the representation

$$\Delta_{\tilde{A}/\tilde{A}'}^\Pi(z) = \varkappa_+ \mathcal{B}_+(z) \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu_+(t) \right\} e^{i\alpha_+ z}, \quad (5.70)$$

$z \in \mathbb{C}_+$. Here $\varkappa_+ \in \mathbb{T}$, $\alpha_+ \geq 0$, ν_+ is a *non-negative Borel measure* satisfying $\int_{\mathbb{R}} (1+t^2)^{-1} d\nu_+(t) < \infty$, and $\mathcal{B}_+(\cdot)$ is the Blaschke product, cf. (D.2), with zeros $\{z_k^+\}_{k \in \mathbb{N}}$. By Proposition 4.8(iv), $\{z_k^+\}_{k \in \mathbb{N}} = \sigma_p(\tilde{A}) \cap \mathbb{C}_+$ and the order m_k^+ of zero z_k^+ equals to the algebraic multiplicity of z_k^+ as the eigenvalue of \tilde{A} .

Next, consider the perturbation determinant $\Delta_{\tilde{A}^*/\tilde{A}'}^\Pi(\cdot)$. Again Lemma 5.12(i) yields the representation

$$\Delta_{\tilde{A}^*/\tilde{A}'}^\Pi(z) = \varkappa_- \mathcal{B}_-(z) \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu_-(t) \right\} e^{i\alpha_- z}, \quad z \in \mathbb{C}_+. \quad (5.71)$$

Here $\mathcal{B}_-(\cdot)$ is the Blaschke product, cf. (D.2), with zeros $\{\zeta_k^+\}_{k \in \mathbb{N}}$ being the eigenvalues of \tilde{A}^* in \mathbb{C}_+ . Moreover, the order n_k^+ of zero ζ_k^+ is equal to the algebraic multiplicity of ζ_k^+ as the eigenvalue of \tilde{A}^* . Note however, that $\zeta_k^+ \in \sigma_p(\tilde{A}^*)$ if and only if $z_k^- := \overline{\zeta_k^+} \in \sigma_p(\tilde{A})$ and the corresponding algebraic multiplicities n_k^+ and m_k^- coincide, $n_k^+ = m_k^-$. Thus, the Blaschke product $\mathcal{B}_-(\cdot)$ is defined by the complex conjugated eigenvalues z_k^- of \tilde{A} lying in \mathbb{C}_- and taken with orders m_k^- equal to their algebraic multiplicities.

Combining (5.70) and (5.71) with the chain rule

$$\Delta_{\tilde{A}/\tilde{A}^*}^\Pi(z) = \frac{\Delta_{\tilde{A}/\tilde{A}'}^\Pi(z)}{\Delta_{\tilde{A}^*/\tilde{A}'}^\Pi(z)}, \quad z \in \rho(\tilde{A}^*) \cap \rho(\tilde{A}) \cap \mathbb{C}_+, \quad (5.72)$$

and setting $c := \frac{c_+}{c_-}$, $\mu := \mu_+ - \mu_-$ and $\alpha := \alpha_+ - \alpha_-$, we arrive at (5.68).

To prove (5.68) for any (not necessarily regular) boundary triplet Π satisfying $\{\tilde{A}^*, \tilde{A}\} \in \mathfrak{D}^\Pi$ it remains to apply Proposition 4.5.

(iv) Taking into account (5.59) we find

$$\begin{aligned} & \operatorname{tr} \left((\tilde{A} - z)^{-1} - (\tilde{A}' - z)^{-1} \right) \\ &= -2i \sum_k \frac{m_k^+ \operatorname{Im}(z_k^+)}{(z - z_k)(z - \bar{z}_k^+)} - \frac{i}{\pi} \int_R \frac{1}{(t - z)^2} d\mu_+(t) - i\alpha_+ \end{aligned} \quad (5.73)$$

for $z \in \rho(\tilde{A}) \cap \mathbb{C}_+$ and

$$\begin{aligned} & \operatorname{tr} \left((\tilde{A}^* - z)^{-1} - (\tilde{A}'^* - z)^{-1} \right) \\ &= -2i \sum_l \frac{m_l^- \operatorname{Im}(\bar{z}_l^-)}{(z - z_l^-)(z - \bar{z}_l^-)} - \frac{i}{\pi} \int_R \frac{1}{(t - z)^2} d\mu_-(t) - i\alpha_- \\ &= 2i \sum_l \frac{m_l^- \operatorname{Im}(z_l^-)}{(z - z_l^-)(z - \bar{z}_l^-)} - \frac{i}{\pi} \int_R \frac{1}{(t - z)^2} d\mu_-(t) - i\alpha_- \end{aligned} \quad (5.74)$$

for $z \in \rho(\tilde{A}^*) \cap \mathbb{C}_+$. Subtracting (5.73) from (5.74) we easily obtain (5.69). \square

Corollary 5.16. *Let the assumptions of Theorem 5.15 be satisfied. If $\Phi \in \mathcal{F}(\tilde{A}, \tilde{A}^*)$, then $\Phi(\tilde{A}) - \Phi(\tilde{A}^*) \in \mathfrak{S}_1(\mathfrak{H})$ and*

$$\begin{aligned} & \operatorname{tr}(\Phi(\tilde{A}) - \Phi(\tilde{A}^*)) = \\ & \sum_n m_n (\Phi(z_n) - \Phi(\bar{z}_n)) + \frac{i}{\pi} \int_{\mathbb{R}} \Phi'(t) d\mu(t) + i\alpha \operatorname{res}_{\infty}(\Phi) \end{aligned} \quad (5.75)$$

where z_n are the eigenvalues of \tilde{A} in $\mathbb{C} \setminus \mathbb{R}$ and m_n their algebraic multiplicities.

Proof. The inclusion $\Phi(\tilde{A}) - \Phi(\tilde{A}^*) \in \mathfrak{S}_1(\mathfrak{H})$ immediately follows from (5.66) and Lemma F.1. Further, we multiply identity (5.69) by $\Phi(\cdot)$ and then integrate the result along a simple closed curve Γ containing the spectra $\sigma(\tilde{A}) \cup \sigma(\tilde{A}^*)$. Applying formulas (F.4), (F.2) and (F.5) we arrive at formula (5.75). \square

Remark 5.17. Corollary 5.16 generalize the known result of V. Adamyan and B. Pavlov [3] and coincide with that in the case of an m -dissipative operator \tilde{A} with $\rho(\tilde{A}) \cap \mathbb{C}_+ \neq \emptyset$. The result has obtained in [3] by applying a functional model of m -dissipative operators [61].

Next we complete and simplify Theorem 5.15 assuming, in addition, that the resolvent of an extension is compact.

Theorem 5.18. *Let the assumptions of Theorem 5.15 be satisfied. If, in addition, $(\tilde{A} - z)^{-1} \in \mathfrak{S}_{\infty}(\mathfrak{H})$, then the following holds:*

(i) *If $\{\tilde{A}, \tilde{A}^*\} \in \mathfrak{D}^{\Pi}$, then the perturbation determinant $\Delta_{\tilde{A}/\tilde{A}^*}^{\Pi}(\cdot)$ is holomorphic in a neighborhood of the real line \mathbb{R} and*

$$|\Delta_{\tilde{A}/\tilde{A}^*}^{\Pi}(x)| = 1 \quad \text{for} \quad x \in \mathbb{R}. \quad (5.76)$$

(ii) If $\{\tilde{A}, \tilde{A}^*\} \in \mathfrak{D}^\Pi$, then the representation (5.68) is simplified to

$$\Delta_{\tilde{A}/\tilde{A}^*}^\Pi(z) = c \frac{\mathcal{B}_+(z)}{\mathcal{B}_-(z)} e^{i\alpha z}, \quad \alpha \in \mathbb{R}, \quad z \in \rho(\tilde{A}) \cap \mathbb{C}_+. \quad (5.77)$$

(iii) The following trace formula holds

$$\mathrm{tr} \left((\tilde{A}^* - z)^{-1} - (\tilde{A} - z)^{-1} \right) = 2i \sum_n \frac{m_n \mathrm{Im}(z_n)}{(z - z_n)(z - \bar{z}_n)} + i\alpha, \quad (5.78)$$

for $z \in \rho(\tilde{A}^*) \cap \rho(\tilde{A})$. In particular, if $a = \bar{a} \in \rho(\tilde{A})$, then

$$\alpha/2 = \mathrm{tr}(\mathrm{Im}(\tilde{A}^* - a)^{-1}) - \sum_n \mathrm{Im} \left(\frac{m_n}{a - z_n} \right) \quad (5.79)$$

where z_n and m_n are the eigenvalues of \tilde{A} in $\mathbb{C} \setminus \mathbb{R}$ and their multiplicities, respectively.

Proof. (i) Let Π be a boundary triplet for A^* regular for $\{\tilde{A}, \tilde{A}^*\}$ such that $\{\tilde{A}, \tilde{A}^*\} \in \mathfrak{D}^\Pi$, cf. Theorem 5.15(i). Since $(\tilde{A}^* - z)^{-1}$ is compact for $z \in \rho(\tilde{A}^*)$ the perturbation determinant $\Delta_{\tilde{A}/\tilde{A}^*}^\Pi(\cdot)$ is meromorphic in \mathbb{C} .

Since Π is regular for $\{\tilde{A}, \tilde{A}^*\}$, one has $\tilde{A} = A_B = A^* \upharpoonright \ker(\Gamma_1 - B\Gamma_0)$ and $\tilde{A}^* = A_{B^*}$ where $B \in [\mathcal{H}]$. Therefore the real part \tilde{A}_R of \tilde{A} is well defined, $\tilde{A}_R := A_{B_R}$. Since $B_I \in \mathfrak{S}_1$, the perturbation determinants $\Delta_{\tilde{A}_R/\tilde{A}}^\Pi(\cdot)$ and $\Delta_{\tilde{A}_R/\tilde{A}^*}^\Pi(\cdot)$ are well defined and

$$\Delta_{\tilde{A}_R/\tilde{A}}^\Pi(z) = \det(I + (B_R - B)(B - M(z))^{-1}) = \det(I - iB_I(B - M(z))^{-1}),$$

$z \in \rho(\tilde{A}) \cap \rho(A_0)$, and

$$\Delta_{\tilde{A}_R/\tilde{A}^*}^\Pi(z) = \det(I + (B_R - B^*)(B^* - M(z))^{-1}) = \det(I + iB_I(B^* - M(z))^{-1}),$$

$z \in \rho(\tilde{A}^*) \cap \rho(A_0)$. Moreover, by Proposition 4.6, the determinants $\Delta_{\tilde{A}_R/\tilde{A}}^\Pi(\cdot)$ and $\Delta_{\tilde{A}_R/\tilde{A}^*}^\Pi(\cdot)$ admit holomorphic continuations from $\rho(\tilde{A}) \cap \rho(\tilde{A}^*) \cap \rho(A_0)$ to $\rho(\tilde{A})$ and $\rho(\tilde{A}^*)$, respectively. Since the resolvents of \tilde{A} and \tilde{A}^* are compact the determinants $\Delta_{\tilde{A}_R/\tilde{A}}^\Pi(\cdot)$ and $\Delta_{\tilde{A}_R/\tilde{A}^*}^\Pi(\cdot)$ are meromorphic. According to (4.14) we get

$$\Delta_{\tilde{A}_R/\tilde{A}^*}^\Pi(z) = \overline{\Delta_{\tilde{A}_R/\tilde{A}}^\Pi(\bar{z})}, \quad z \in \rho(\tilde{A}^*).$$

In particular, we have

$$\Delta_{\tilde{A}_R/\tilde{A}^*}(x) = \overline{\Delta_{\tilde{A}_R/\tilde{A}}(x)}, \quad x \in \rho(\tilde{A}) \cap \rho(\tilde{A}^*) \cap \mathbb{R} = \rho(\tilde{A}^*) \cap \mathbb{R} = \rho(\tilde{A}) \cap \mathbb{R}.$$

Using this identity and applying the chain rule we get

$$\Delta_{\tilde{A}/\tilde{A}^*}(x) = \frac{\Delta_{\tilde{A}_R/\tilde{A}^*}(x)}{\Delta_{\tilde{A}_R/\tilde{A}}(x)}, \quad x \in \rho(\tilde{A}) \cap \rho(\tilde{A}^*) \cap \rho(\tilde{A}_R) \cap \mathbb{R}. \quad (5.80)$$

It follows that $|\Delta_{\tilde{A}/\tilde{A}^*}(x)| = 1$ for $x \in \rho(\tilde{A}) \cap \rho(\tilde{A}_R) \cap \mathbb{R}$. Since $(\tilde{A}_R - z)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H})$, $z \in \rho(\tilde{A})$, the operator \tilde{A}_R has also discrete spectrum, $\sigma(\tilde{A}_R) = \sigma_d(\tilde{A}_R)$. Thus, $|\Delta_{\tilde{A}/\tilde{A}^*}(x)| = 1$ for x outside a discrete set $(\sigma(\tilde{A}_R) \cup \sigma(\tilde{A})) \cap \mathbb{R}$. Hence any possible real pole x_0 of the meromorphic function $\Delta_{\tilde{A}/\tilde{A}^*}^\Pi(\cdot)$ is removable. Thus, $|\Delta_{\tilde{A}/\tilde{A}^*}(x)| = 1$ for any $x \in \mathbb{R}$ which shows that $\Delta_{\tilde{A}/\tilde{A}^*}(\cdot)$ is holomorphic in a neighborhood of \mathbb{R} .

(ii) Clearly, the extension $\tilde{A}' = A_{B_1^*}$, $B_1 := B_R + i|B_I|$, is m -accumulative. Moreover, since $B - B_1^* = 2i|B_I| \in \mathfrak{S}_1(\mathcal{H})$, it follows from Proposition 2.6(ii) that the resolvent of \tilde{A}' is compact, i.e., the spectrum of \tilde{A}' is discrete. Hence, the perturbation determinant $F_+(\cdot) := \Delta_{\tilde{A}/\tilde{A}'}^\Pi(\cdot)$ is holomorphic in \mathbb{C}_+ and meromorphic in \mathbb{C} . In particular, $F_+(\cdot)$ admits a holomorphic continuation through $\mathbb{R} \setminus \sigma(\tilde{A}') = \mathbb{R} \setminus \sigma_p(\tilde{A}')$ where, of course, $\sigma_p(\tilde{A}') \cap \mathbb{R}$ is a discrete set. From [22, Theorem II.6.3] we find that the inner and outer factors $I_{F_+}(\cdot)$ and $\mathcal{O}_{F_+}(\cdot)$, respectively, of the contractive in \mathbb{C}_+ holomorphic function $F_+(\cdot) := \Delta_{\tilde{A}/\tilde{A}'}^\Pi(\cdot)$, cf. Appendix D, admit also a holomorphic continuation through $\mathbb{R} \setminus \sigma(\tilde{A}')$. Since the Blaschke product $\mathcal{B}_+(\cdot)$ admits a holomorphic continuation through $\mathbb{R} \setminus \sigma_p(\tilde{A}')$ we get that the singular factor $S_{F_+}(\cdot)$ (cf. (D.5)) has this property. By [22, Theorem II.6.2] the singular part μ_+^s of the measure μ_+ is supported on $\mathbb{R} \cap \sigma(\tilde{A}')$. Thus, the singular continuous part μ_+^{sc} of the measure μ_+ is missing, i.e. $\mu_+^{sc} \equiv 0$. Hence, μ_+^s is atomic and supported on $\sigma(\tilde{A}')$, i.e.

$$S_{F_+}(x) = \exp \left\{ \frac{i}{\pi} \sum_{t_k \in \sigma(\tilde{A}') \cap \mathbb{R}} \left(\frac{1}{t_k - x} - \frac{t_k}{1 + t_k^2} \right) \mu_+^s(\{t_k\}) \right\}, \quad (5.81)$$

$x \in \mathbb{R} \setminus \sigma(\tilde{A}')$. By a straightforward computations it follows from (D.3) that

$$\lim_{y \rightarrow +0} |\Delta_{\tilde{A}/\tilde{A}'}^\Pi(x + iy)| = e^{-\mu_+'(x)} \quad \text{for a.e. } x \in \mathbb{R}. \quad (5.82)$$

By the same reasoning we get that the singular factor $S_{F_-}(\cdot)$ of $F_-(\cdot) := \Delta_{\tilde{A}^*}/\tilde{A}'(\cdot)$, (cf. (D.5)) admits the representation

$$S_{F_-}(x) = \exp \left\{ \frac{i}{\pi} \sum_{t_k \in \sigma(\tilde{A}') \cap \mathbb{R}} \left(\frac{1}{t_k - x} - \frac{t_k}{1 + t_k^2} \right) \mu_-^s(\{t_k\}) \right\}, \quad (5.83)$$

$x \in \mathbb{R} \setminus \sigma(\tilde{A}')$, and $\mu_-^{sc} \equiv 0$. Moreover, it follows from (5.71) that

$$\lim_{y \rightarrow +0} |\Delta_{\tilde{A}^*}/\tilde{A}'^\Pi(x + iy)| = e^{-\mu_-'(x)} \quad \text{for a.e. } x \in \mathbb{R}. \quad (5.84)$$

Combining (5.80) with (5.82) and (5.84) we get

$$|\Delta_{\tilde{A}/\tilde{A}'}^\Pi(x)| = e^{-(\mu_+'(x) - \mu_-'(x))} \quad \text{for a.e. } x \in \mathbb{R}.$$

Since $|\Delta_{\tilde{A}/\tilde{A}'}^\Pi(x)| = 1$ for $x \in \mathbb{R}$ we get $\mu'_+(x) = \mu'_-(x)$ for a.e. $x \in \mathbb{R}$, i.e. $\mu_+^{ac} = \mu_-^{ac}$. Hence $\mathcal{O}_{F_+}(z) = \mathcal{O}_{F_-}(z)$, $z \in \mathbb{C}_+$ which yields the representation

$$\Delta_{\tilde{A}/\tilde{A}'}^\Pi(z) = \frac{F_+(z)}{F_-(z)} = \frac{I_{F_+}(z)}{I_{F_-}(z)} = \frac{\varkappa_+ \mathcal{B}_+(z) S_{F_+}(z)}{\varkappa_- \mathcal{B}_-(z) S_{F_-}(z)} e^{i(\alpha_+ - \alpha_-)}, \quad z \in \rho(\tilde{A}^*) \cap \mathbb{C}_+.$$

Since the spectrum of \tilde{A} is discrete the eigenvalues z_k^+ of \tilde{A} in \mathbb{C}_+ cannot accumulate to the real axis. Hence the limit $\mathcal{B}_+(x) = \lim_{y \rightarrow +0} \mathcal{B}_+(x + iy)$ exists for any $x \in \mathbb{R}$ and the limit function $\mathcal{B}_+(x)$ is continuous. Moreover, $|\mathcal{B}_+(x)| = 1$ for $x \in \mathbb{R}$. Similarly one shows that the limit function $\mathcal{B}_-(x)$ is defined everywhere, is continuous and $|\mathcal{B}_-(x)| = 1$ for $x \in \mathbb{R}$. Hence $\frac{\mathcal{B}_+(x)}{\mathcal{B}_-(x)}$ is continuous.

Therefore the limit function $S(x) := \lim_{y \rightarrow +0} \frac{S_{F_+}(x + iy)}{S_{F_-}(x + iy)}$ exists everywhere and is continuous. Combining (5.81) with (5.83) we get the representation

$$S(x) = \exp \left\{ \frac{i}{\pi} \sum_{t_k \in \sigma(\tilde{A}') \cap \mathbb{R}} \left(\frac{1}{t_k - x} - \frac{t_k}{1 + t_k^2} \right) (\mu_+^s(\{t_k\}) - \mu_-^s(\{t_k\})) \right\}$$

for $x \in \mathbb{R} \setminus \sigma(\tilde{A}')$. It is easily seen that the function $S(\cdot)$ is continuous at t_k if and only if $\mu_+^s(\{t_k\}) = \mu_-^s(\{t_k\})$, $t_k \in \sigma(\tilde{A}') \cap \mathbb{R}$ which yields $S(x) = 1$ for $x \in \mathbb{R}$. Thus, we arrive at the representation (5.77) with $c := \frac{\varkappa_+}{\varkappa_-}$ and $\alpha := \alpha_+ - \alpha_-$.

To prove (5.77) for any (not necessarily regular) boundary triplet Π satisfying $\{\tilde{A}^*, \tilde{A}\} \in \mathfrak{D}^\Pi$ it remains to apply Proposition 4.5.

(iii) Formula (5.78) follows from (5.69) with the measure $\mu = 0$. Formula (5.79) follows immediately from (5.78). \square

Corollary 5.19. *Let the assumptions of Theorem 5.18 be satisfied. If $\Phi \in \mathcal{F}(\tilde{A}, \tilde{A}^*)$, then $\Phi(\tilde{A}^*) - \Phi(\tilde{A}) \in \mathfrak{S}_1(\mathfrak{H})$ and*

$$\text{tr}(\Phi(\tilde{A}) - \Phi(\tilde{A}^*)) = \sum_n m_n (\Phi(z_n) - \Phi(\bar{z}_n)) + i\alpha \text{res}_\infty(\Phi)$$

where $\{z_n\}_{n \in \mathbb{R}}$ are the non-real eigenvalues of \tilde{A} and m_n their algebraic multiplicities, respectively.

Proof. Corollary 5.19 follows from Corollary 5.16 setting $\mu \equiv 0$. \square

Remark 5.20.

(i) If \tilde{A} is m -dissipative, then the perturbation determinant $\Delta_{\tilde{A}/\tilde{A}^*}(\cdot)$ is holomorphic and contractive in \mathbb{C}_+ and due to (5.76) it is an inner function in \mathbb{C}_+ .

In contrast to this fact, in the non-dissipative case the perturbation determinant $\Delta_{\tilde{A}/\tilde{A}^*}^\Pi(\cdot)$ admits the representation $\Delta_{\tilde{A}/\tilde{A}^*}^\Pi(z) = \frac{F_+(z)}{F_-(z)}$, $z \in \mathbb{C}_+$, where the numerator and the denominator might really have outer factors despite of the analyticity of both determinants on the real line and the necessary condition $|\Delta_{\tilde{A}/\tilde{A}^*}^\Pi(x)| = 1$ for $x \in \mathbb{R}$ (cf. (5.76)).

(ii) Notice that the non-dissipative operator \tilde{A} might have real eigenvalues even if it is completely non-selfadjoint. However these eigenvalues do not appear in the representation (5.77) neither in the trace formula (5.78). This fact is not surprising since if $\lambda_0 = \bar{\lambda}_0 \in \sigma_p(\tilde{A})$ then $\lambda_0 \in \sigma_p(\tilde{A}^*)$ and $\dim \ker(\tilde{A} - \lambda_0) = \dim \ker(\tilde{A}^* - \lambda_0)$ and these zeros cancel out in the representation (5.77). Due to formula (5.67) such eigenvalues do not appear in the determinant of the characteristic function $W_{\tilde{A}}^{\Pi}(\cdot)$. In this connection we mention the paper [62] where it is shown that even singular factors cancel in a formula for the determinant of the characteristic function.

Theorem 5.18 allows to indicate a condition which guarantees the completeness of the root vector system, cf. Appendix E.

Corollary 5.21. *Let the assumption of Theorem 5.18 be satisfied and let us assume in addition that \tilde{A} is a maximal dissipative operator. The root vector system of \tilde{A} is complete if and only if $\alpha = 0$.*

Proof. Let \tilde{A} be a maximal dissipative extension of A such the $\rho(\tilde{A}) \neq \emptyset$. Since $(\tilde{A} - z)^{-1}$ is compact for some $z \in \rho(\tilde{A})$ there is a real number $a \in \rho(\tilde{A})$. Hence $a \in \rho(\tilde{A}^*)$. Let $R := (\tilde{A}^* - a)^{-1}$. A simple computation shows that R is a bounded dissipative operator. From (5.66) one gets that $(\tilde{A}^* - a)^{-1} - (\tilde{A} - a)^{-1}$ is a trace class operator. Hence $R - R^*$ is trace class operator which yields $R_I := \text{Im}(R)$ is a trace class operator. Since R is dissipative we obtain from (5.79) that

$$\text{tr}(R_I) = \sum_n m_n \text{Im}(\mu_n) + \frac{\alpha}{2}, \quad \mu_n := \frac{1}{\bar{z}_n - a},$$

where z_n and m_n the eigenvalues of \tilde{A} in \mathbb{C}_+ and their multiplicities. Since \tilde{A} is maximal dissipative it holds $\alpha \geq 0$. We note that μ_n are the eigenvalues of T and m_n their multiplicities. Applying Theorem V.2.1 of [32] we prove that the root vector system of R is complete if and only if $\alpha = 0$. Using Lemma G.1 we get that root vector system of \tilde{A} is complete if and only if $\alpha = 0$. \square

5.5 Annihilation functions for dissipative extensions

We are going to prove a Cayley-Hamiltonian-type theorem for maximal dissipative extensions of a symmetric operator A with finite deficiency indices. Let A be a densely defined closed symmetric operator. A point $z \in \mathbb{C}$ is called of regular type of A , cf. [1, Section VIII.100], if there is a constant $c > 0$ such that $\|(A - z)f\|^2 \geq c\|f\|^2$, $f \in \text{dom}(A)$. By $\hat{\rho}(A)$ we denote the set of all points of regular type of A .

Proposition 5.22. *Let A be a simple closed symmetric operator in \mathfrak{H} with finite deficiency indices $n := n_+(A) = n_-(A) < \infty$ and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* . Further, let $\tilde{A} \in \text{Ext}_A$ be a maximal dissipative extension of A such that $\tilde{A} = A_B$ with $B \in [\mathcal{H}]$. Assume that $\hat{\rho}(A) = \mathbb{C}$. Then the following holds:*

- (i) The resolvent of \tilde{A} is compact, i.e. the spectrum of \tilde{A} is discrete.
- (ii) If $\ker(B_I) = \{0\}$, $B_I = \operatorname{Im}(B)$, then \tilde{A} is completely non-selfadjoint. In particular, $\mathbb{R} \subset \rho(\tilde{A})$.
- (iii) If \tilde{A} is completely non-selfadjoint, then \tilde{A} belongs to the class C_0 . Moreover, the perturbation determinant $d(z) := \Delta_{\tilde{A}/\tilde{A}^*}^{\Pi}(z)$, $z \in \mathbb{C}_+$, is an annihilation function for \tilde{A} .
- (iv) If \tilde{A} is completely non-selfadjoint and complete, then the annihilation function $d(\cdot)$ is minimal for \tilde{A} if and only if the geometric multiplicity of any eigenvalue z of \tilde{A} is one, i.e. $\dim(\ker(\tilde{A} - z)) = 1$ or, equivalently, $\dim(\ker(B - M(z))) = 1$. In particular, $d(\cdot)$ is minimal if $n_{\pm}(A) = 1$.

Proof. (i) Follow [1, Section 105] $z \in \mathbb{C}$ belongs to the continuous spectrum of A if $z \in \mathbb{C} \setminus \hat{\rho}(A)$ and $\operatorname{ran}(A - z)$ is not closed. By [1, Theorem 100.1] all selfadjoint extension of A have the same continuous spectrum. Since $\hat{\rho}(A) = \mathbb{C}$ we find that for all selfadjoint extensions the continuous spectrum is empty. Hence, for any selfadjoint extension the continuous spectrum is discrete which shows that the resolvent of any selfadjoint extension of A is compact. Applying Krein formula (2.7) we find that the resolvent of any other extension is compact, too.

(ii) Let us show that \tilde{A} is completely non-selfadjoint. From Proposition 2.3 it follows that the operator B has to be dissipative, i.e. $\operatorname{Im}(B) \geq 0$.

Since the spectrum of \tilde{A} is discrete, it suffices to show that \tilde{A} has no real eigenvalues. Let us assume that $x \in \sigma(\tilde{A}) \cap \mathbb{R}$. It follows from the Green formula (2.1) that for any $x \in \mathbb{R}$ the following identity holds

$$\begin{aligned} \operatorname{Im}((A_B - x)f, f) &= -i[(B\Gamma_0 f, \Gamma_0 f)_{\mathcal{H}} - (\Gamma_0 f, B\Gamma_0 f)_{\mathcal{H}}] \\ &= 2(B_I \Gamma_0 f, \Gamma_0 f)_{\mathcal{H}} = 2 \left\| \sqrt{B_I} \Gamma_0 f \right\|_{\mathcal{H}}^2, \quad f \in \operatorname{dom}(A^*). \end{aligned}$$

Let $f \in \ker(A_B - x)$. Since $\ker(B_I) = \{0\}$, the above identity yields $\Gamma_0 f = 0$ and $\Gamma_1 f = B\Gamma_0 f = 0$. Thus, $f \in \operatorname{dom}(A)$ and $f \in \ker(A - x)$. This contradicts the simplicity of A . Thus, $\mathbb{R} \subset \rho(\tilde{A})$.

(iii) Since A has finite deficiency indices, $\{\tilde{A}, \tilde{A}^*\} \in \mathfrak{D}^{\Pi}$. Thus, the perturbation determinant $d(\cdot) := \Delta_{\tilde{A}/\tilde{A}^*}^{\Pi}(\cdot)$ exists on $\rho(\tilde{A}^*)$. By Theorem 5.15(ii), $d(z) = \det(W_{\tilde{A}}(z))$, $z \in \rho(\tilde{A}^*) \cap \rho(A_0)$ where $W_{\tilde{A}}(\cdot)$ is the characteristic function of \tilde{A} defined by (3.3) with $J = I$.

Since \tilde{A} is m -dissipative, the characteristic function $W_{\tilde{A}}(\cdot)$ is contractive in \mathbb{C}_+ . Hence, $d(\cdot)$ is contractive in \mathbb{C}_+ . Further, since, by (i), the resolvent of \tilde{A} is compact, it follows from Theorem 5.18(ii) that $d(\cdot)$ is an inner function. Hence $W_{\tilde{A}}(\cdot)$ is an inner operator-valued function (see [61, Corollary V.6.3]). Applying [61, Proposition VI.3.5] we obtain that \tilde{A} belongs to the class C_0 .

Similarly, since the operator $-\tilde{A}^*$ is m -dissipative too, its characteristic function $W_{-\tilde{A}^*}(\cdot)$ is also inner operator-valued function in \mathbb{C}_+ . Hence $-\tilde{A}^*$

belongs to the class $C_{\cdot 0}$ which is equivalent to the inclusion $\tilde{A} \in C_0$. Therefore $\tilde{A} \in C_{\cdot 0} \cap C_0 = C_{00}$.

Since $n_{\pm}(A) = n < \infty$, the contraction $T_{\tilde{A}} = (\tilde{A} - i)(\tilde{A} + i)^{-1}$ (cf. Appendix H), has equal finite defect numbers, i.e. $\dim \left(\text{ran}(I - T_{\tilde{A}}^* T_{\tilde{A}}) \right) = \dim \left(\text{ran}(I - T_{\tilde{A}} T_{\tilde{A}}^*) \right) < \infty$. By [61, Theorem VI.5.2], $T_{\tilde{A}} \in C_0$ and the determinant $d(\cdot) = \det(W_{\tilde{A}}^{\Pi}(\cdot))$ defined on \mathbb{C}_+ , is an annihilation function for \tilde{A} , i.e. $d(\tilde{A}) = \tilde{d}(T_{\tilde{A}}) = 0$, cf. Appendix H.

(iv) Let $\{e_j\}_{j=1}^n$ be a fixed orthonormal basis in \mathcal{H} . Denote by $\Theta_{\tilde{A}}(\cdot)$ the matrix representation of the characteristic function $W_{\tilde{A}}^{\Pi}(\cdot)$ with respect to the basis $\{e_j\}_{j=1}^n$. By $\text{adj}(\Theta_{\tilde{A}}(\cdot))$ we denote the adjugate matrix of $\Theta_{\tilde{A}}(\cdot)$. Note that alongside the matrix $\Theta_{\tilde{A}}(\cdot)$ the adjugate matrix function $\text{adj}(\Theta_{\tilde{A}}(z))$ is holomorphic and contractive in \mathbb{C}_+ too (cf. the proof of [61, Proposition V.6.1]). By [61, Theorem VI.5.2], the determinant $d(\cdot) := \det(\Theta_{\tilde{A}}(\cdot)) = \det(W_{\tilde{A}}^{\Pi}(\cdot))$ of $\Theta_{\tilde{A}}(\cdot)$ coincides with the minimal annihilation function $m_{\tilde{A}}(\cdot)$ of \tilde{A} if and only if the entries of $\text{adj}(\Theta_{\tilde{A}}(\cdot))$ have no common non-trivial inner divisor in the algebra $H^{\infty}(\mathbb{C}_+)$.

On the other hand, by (iii), the operator $\tilde{A} \in C_0$. Therefore it is complete if and only if the determinant $d(\cdot)$ is a Blaschke product (see [53, Section 4.5]). Therefore it follows from the identity $\text{adj}(\Theta_{\tilde{A}}(z)) \cdot \Theta_{\tilde{A}}(z) = d(z)I_n$ that each common divisor $\varphi(\cdot)$ of the entries of $\text{adj}(\Theta_{\tilde{A}}(\cdot))$ has to be a divisor of $d(\cdot)$. Therefore $\varphi(\cdot)$ always contains a Blaschke factor, i.e. it admits the representation $\varphi(\cdot) = \varphi_1(\cdot)b_{z_0}^{m_0}(\cdot)$ where $b_{z_0}^{m_0}(\cdot)$ is a Blaschke factor $b_{z_0}^{m_0}(z) := (e^{i\alpha_0}(z - z_0)/(z - \bar{z}_0))^{m_0}$, $m_0 \geq 1$, $z_0 \in \mathbb{C}_+$, cf. (D.2). Clearly, the latter happens if and only if $\text{adj}(\Theta_{\tilde{A}}(z_0)) = 0_n := 0 \cdot I_n$. However, $\text{adj}(\Theta_{\tilde{A}}(z_0)) = 0_n$ is valid if and only if $\text{rank}(\Theta_{\tilde{A}}(z_0)) \leq n - 2$, that is, $\dim(\ker(\Theta_{\tilde{A}}(z_0))) = \dim(\ker(W_{\tilde{A}}^{\Pi}(z_0))) \geq 2$.

Further, by Proposition 2.5(ii), $\dim \ker(\tilde{A} - z) = \dim \ker(B - M(z))$ for any $z \in \rho(A_0)$. Let us show that

$$\dim(\ker(W_{\tilde{A}}^{\Pi}(z_0))) = \dim(\ker(B - M(z_0))), \quad z_0 \in \mathbb{C}_+. \quad (5.85)$$

Indeed, setting $T_1 := 2i(B^* - M(z_0))^{-1}B_I^{1/2}$ and using $\ker(B_I) = \{0\}$ one immediately verifies that $\ker(T_1) = \{0\}$. Further, we note that $W_{A_B}(z_0)h_0 = 0$ if and only if $h_0 \in \ker(I + B_I^{1/2}T_1)$. If $h_0 \in \ker(I + B_I^{1/2}T_1)$, then $T_1h_0 \in \ker(I + T_1B_I^{1/2})$ which yields $\dim(\ker(I + B_I^{1/2}T_1)) \leq \dim(\ker(I + T_1B_I^{1/2}))$. Conversely, if $h_1 \in \ker(I + T_1B_I^{1/2})$, then $B_I^{1/2}h_1 \in \ker(I + B_I^{1/2}T_1)$ which proves $\dim(\ker(I + T_1B_I^{1/2})) \leq \dim(\ker(I + B_I^{1/2}T_1))$. Hence

$$\dim(\ker(W_{\tilde{A}}(z_0))) = \dim(\ker(I + B_I^{1/2}T_1)) = \dim(\ker(I + T_1B_I^{1/2})).$$

Combining this relation with the identity $I + T_1B_I^{1/2} = (B^* - M(z_0))^{-1}(B - M(z_0))$ we arrive at (5.85). Thus, $d(\cdot) = \Delta_{\tilde{A}/\tilde{A}^*}^{\Pi}(\cdot)$ is a minimal annihilation

function if and only if $\dim(\ker(B - M(z))) = 1$ for $z \in \sigma(\tilde{A})$ which yields $\dim(\ker(\tilde{A} - z)) = 1$. \square

6 The case of additive perturbations

Here we extend some results of Section 5 for extensions to the case of additive trace class perturbations of m -accumulative operators. We emphasize that all results of this section are new for the case additive perturbations.

6.1 The pairs of m -accumulative operators

We start with two technical statements.

Lemma 6.1. *Assume that H and H' are maximal accumulative operators in \mathfrak{H} and $V \in \mathfrak{S}_1(\mathfrak{H})$, Then*

$$\lim_{y \rightarrow \infty} y^2 \operatorname{tr}((H' - iy)^{-1} V (H - iy)^{-1}) = -\operatorname{tr}(V). \quad (6.1)$$

Proof. The proof is based on the following statement: Let $\{Z(y)\}_{y \in \mathbb{R}_+}$ be a family of bounded operators such that $\operatorname{s-lim}_{y \rightarrow \infty} Z(y) = Z$. If $V \in \mathfrak{S}_1$, then $\tilde{Z}(y) := Z(y)V \in \mathfrak{S}_1(\mathfrak{H})$, $y \in \mathbb{R}_+$, tends to ZV in the \mathfrak{S}_1 -norm, see [32, Theorem III.6.3].

Let $Z(y) := y^2(H - iy)^{-1}(H' - iy)^{-1}$, $y \in \mathbb{R}_+$. Since H and H' are m -accumulative,

$$\operatorname{s-lim}_{y \rightarrow \infty} y(H - iy)^{-1} = iI \quad \text{and} \quad \operatorname{s-lim}_{y \rightarrow \infty} y(H' - iy)^{-1} = iI,$$

which yields $\operatorname{s-lim}_{y \rightarrow \infty} Z(y) = -I$. Applying the statement above we get $\lim_{y \rightarrow \infty} \|Z(y)V + V\|_{\mathfrak{S}_1} = 0$. Hence

$$\lim_{y \rightarrow \infty} y^2 \operatorname{tr}((H' - iy)^{-1} V (H - iy)^{-1}) = \lim_{y \rightarrow \infty} \operatorname{tr}(Z(y)V) = -\operatorname{tr}(V)$$

which proves (6.1). \square

Corollary 6.2. *Let $V \in \mathfrak{S}_1(\mathfrak{H})$ and let H be maximal accumulative in \mathfrak{H} . Let also $V_I := \operatorname{Im}(V) = V_I^+ - V_I^-$ where $V_I^\pm \geq 0$. If $H' := H + V$, then $z = x + iy \in \rho(H')$ for $y > \|V_I^+\|$ and (6.1) holds.*

Proof. We note that the operator $H' - i\|V_I^+\|$ is accumulative. Using the representation $H' - iy = H' - i\|V_I^+\| - i(y - \|V_I^+\|)$ we find that $i(y - \|V_I^+\|) \in \rho(H' - \|V_I^+\|)$ provided that $y - \|V_I^+\| > 0$. Since $i(y - \|V_I^+\|) \in \rho(H)$ it remains to apply Lemma 6.1. \square

Next we present a counterpart to Lemma 5.2 for additive perturbations.

Lemma 6.3. *Let H be a maximal accumulative operator.*

(i) If $0 \leq V_+ = V_+^* \in \mathfrak{S}_1(\mathfrak{H})$, then there exists a non-negative function $\xi_+(\cdot) \in L^1(\mathbb{R})$ such that the representation

$$\det(I + V_+(H - z)^{-1}) = \exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \frac{\xi_+(t)}{t - z} dt \right\}, \quad z \in \mathbb{C}_+, \quad (6.2)$$

holds and $\text{tr}(V_+) = \frac{1}{\pi} \int_{\mathbb{R}} \xi_+(t) dt$ is satisfied.

(ii) If $V = V^* \in \mathfrak{S}_1(\mathfrak{H})$, then there exists a real-valued function $\xi(\cdot) \in L^1(\mathbb{R})$ such that the representation (6.2) is valid with V and $\xi(\cdot)$ in place of V_+ and $\xi_+(\cdot)$, in particular,

$$\text{tr}(V) = \int_{\mathbb{R}} \xi(t) dt \quad \text{and} \quad \int_{\mathbb{R}} |\xi(t)| dt \leq \|V\|_{\mathfrak{S}_1}. \quad (6.3)$$

Proof. (i) Let $V = V_+ \geq 0$. We mimic the proof of Lemma 5.2(i) replacing B and $M(z)$ by H and z , respectively. Doing so we find a non-negative function $\xi_+(\cdot)$ satisfying $\int \frac{1}{1+t^2} \xi_+(t) dt < \infty$ and a positive constant c_+ such that the representation

$$\det(I + V_+(H - z)^{-1}) = c_+ \exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) \xi_+(t) dt \right\}, \quad (6.4)$$

$z \in \mathbb{C}_+$, is valid. Setting $T(z) := \sqrt{V_+}(H - z)^{-1}\sqrt{V_+}$, $z \in \mathbb{C}_+$, we define a family of dissipative operators. Clearly, $\lim_{y \rightarrow \infty} \|T(x + iy)\| = 0$, $x \in \mathbb{R}$. Hence, $0 \in \rho(I + T(x + iy))$ for any fixed $x \in \mathbb{R}$ and sufficiently large $y > 0$. Thus, for y large enough we can take logarithm of both sides of (6.4) using definition (C.1),

$$\log \det(I + V_+(H - z)^{-1}) = \log(c_+) + \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) \xi_+(t) dt, \quad (6.5)$$

$z \in \mathbb{C}_+$. Hence

$$\text{Im}(\log \det(I + T(z))) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(t - x)^2 + y^2} \xi_+(t) dt, \quad z = x + iy \in \mathbb{C}_+.$$

Using (C.3) we obtain

$$\text{Im}(\text{tr}(\log(I + T(z)))) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(t - x)^2 + y^2} \xi_+(t) dt, \quad z = x + iy \in \mathbb{C}_+. \quad (6.6)$$

It is easily seen that $s - \lim_{y \uparrow \infty} (H - x - iy)^{-1} = 0$ and $s - \lim_{y \uparrow \infty} (-iy)(H - x - iy)^{-1} = I$. Since $V_+ \in \mathfrak{S}_1(\mathfrak{H})$, it follows with account of [32, Theorem 3.6.3] that

$$\lim_{y \uparrow \infty} \|T(x + iy)\|_{\mathfrak{S}_1} = 0 \quad \text{and} \quad \lim_{y \uparrow \infty} \|(-iy)T(x + iy) - V_+\|_{\mathfrak{S}_1} = 0.$$

Combining these relations with definition (C.2) we obtain

$$\begin{aligned} & \lim_{y \uparrow \infty} y \log(I + T(x + iy)) \\ &= - \lim_{y \uparrow \infty} (-iy)T(x + iy) \lim_{y \uparrow \infty} \int_{\mathbb{R}_+} (I + T(x + iy) + i\lambda)^{-1} (1 + i\lambda)^{-1} d\lambda \\ &= -V_+ \int_{\mathbb{R}_+} (1 + i\lambda)^{-2} d\lambda = iV_+ \end{aligned}$$

for any fixed $x \in \mathbb{R}$. Notice that the convergence takes place in the \mathfrak{S}_1 -norm. Hence for any fixed $x \in \mathbb{R}$

$$\lim_{y \rightarrow \infty} y \operatorname{Im} (\operatorname{tr} (\log(I + T(x + iy)))) = \operatorname{tr} (V_+). \quad (6.7)$$

On the other hand, multiplying the identity (6.6) by y and tending y to $+\infty$ we arrive at the equality (6.7) with $\frac{1}{\pi} \int_{\mathbb{R}} \xi_+(t) dt$ in place of $\operatorname{tr} (V_+)$. Thus, these two quantities equal, $\frac{1}{\pi} \int_{\mathbb{R}} \xi_+(t) dt = \operatorname{tr} (V_+)$, in particular, $\xi_+(\cdot) \in L^1(\mathbb{R})$. Combining the later inclusion with representation (6.5) yields the representation

$$\det(I + V_+(H - z)^{-1}) = c'_+ \exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \frac{\xi_+(t)}{t - z} dt \right\}, \quad z \in \mathbb{C}_+, \quad (6.8)$$

where

$$c'_+ := c_+ \exp \left\{ -\frac{1}{\pi} \int_{\mathbb{R}} \frac{t}{1 + t^2} \xi_+(t) dt \right\}.$$

Setting in (6.8) $z = iy$ and tending y to $+\infty$ and noting that $\lim_{y \rightarrow \infty} \det(I + V_+(H - iy)^{-1}) = 1$ we find $c'_+ = 1$ which proves (6.2).

(ii) Setting $K := H - V_-$ where $V = V_+ - V_-$, $V_{\pm} \geq 0$, and using the chain rule for perturbation determinants we get

$$\det(I + V(H - z)^{-1}) = \frac{\det(I + V_+(K - z)^{-1})}{\det(I + V_-(K - z)^{-1})}, \quad z \in \mathbb{C}_+. \quad (6.9)$$

Applying (i) we get the representation (6.2) with non-negative $\xi_+(\cdot) \in L^1(\mathbb{R})$ and a similar representation with non-negative $\xi_-(\cdot) \in L^1(\mathbb{R})$ for $\det(I + V_-(K - z)^{-1})$. Setting $\xi := \xi_+(t) - \xi_-(t)$, $t \in \mathbb{R}$ and using (6.9) we arrive at the representation (6.2) for $\det(I + V(H - z)^{-1})$, $z \in \mathbb{C}_+$. Further, since $\frac{1}{\pi} \int_{\mathbb{R}} \xi_{\pm}(t) dt = \operatorname{tr} (V_{\pm})$, we get $\frac{1}{\pi} \int_{\mathbb{R}} \xi(t) dt = \operatorname{tr} (V)$. Moreover,

$$\frac{1}{\pi} \int_{\mathbb{R}} |\xi(t)| dt \leq \frac{1}{\pi} \int_{\mathbb{R}} \xi_+(t) dt + \frac{1}{\pi} \int_{\mathbb{R}} \xi_-(t) dt = \operatorname{tr} (V_+) + \operatorname{tr} (V_-) = \|V\|_{\mathfrak{S}_1},$$

which proves (6.3). \square

Remark 6.4. Lemma 6.3 can be proved in a quite different way using the classical results of Krein in [41, 44], see also [45] and [11]. Indeed, since H is a maximal accumulative operator it admits a selfadjoint dilation, that is, there is a selfadjoint operator K in a larger Hilbert space $\mathfrak{K} \supseteq \mathfrak{H}$ such that

$$(H - z)^{-1} = P_{\mathfrak{H}}^{\mathfrak{K}} (K - z)^{-1} \upharpoonright \mathfrak{H}, \quad z \in \mathbb{C}_+.$$

cf. [61]. Notice that

$$\Delta_{H'/H}(z) = \det(I_{\mathfrak{H}} + V(H - z)^{-1}) = \det(I_{\mathfrak{K}} + V(K - z)^{-1}) = \Delta_{K'/K}(z),$$

$z \in \mathbb{C}_+$, where $H' := H + V$ and $K' := K + V$. By Theorem 1 of [44] we immediately find a real-valued function $\xi(\cdot) \in L^1(\mathbb{R})$ such that statement (ii) of Lemma 6.3 is valid, in particular, the relations (6.3) hold. Moreover, If $V \geq 0$, then the same theorem guarantees the existence of a non-negative function $\xi(\cdot) \in L^1(\mathbb{R})$ such that (i) of Lemma 6.3 is valid.

Using Lemma 6.3 one readily verifies trace formula (5.1). Passing to m -accumulative operators H we firstly prove an additive counterpart of Lemma 5.6.

Lemma 6.5. *Let H be a maximal accumulative operator in \mathfrak{H} .*

(i) *Let $0 \leq V_+ = V_+^* \in \mathfrak{S}_1(\mathfrak{H})$. If the condition*

$$(V_+ f, f) \leq -\operatorname{Im}(H f, f), \quad f \in \operatorname{dom}(H). \quad (6.10)$$

is satisfied, then the function $w_+(z) := \det(I + iV_+(H - z)^{-1})$ admits the representation

$$w_+(z) = \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \frac{1}{t - z} \eta_+(t) dt \right\}, \quad z \in \mathbb{C}_+, \quad (6.11)$$

with non-negative $\eta_+(\cdot) \in L^1(\mathbb{R})$. Moreover, the representation $\eta_+(t) = -\ln(|w_+(t + i0)|)$ holds for a.e. $t \in \mathbb{R}$ where $w_+(t + i0) := \lim_{y \downarrow 0} w_+(t + iy)$.

In addition, the function $w^+(z) := \det(I - iV_+(H - z)^{-1})$, $z \in \mathbb{C}_+$, admits the representation

$$w^+(z) = \exp \left\{ -\frac{i}{\pi} \int_{\mathbb{R}} \frac{1}{t - z} \eta^+(t) dt \right\}, \quad z \in \mathbb{C}_+, \quad (6.12)$$

with non-negative $\eta^+(\cdot) \in L^1(\mathbb{R})$ such that $\eta^+(t) = \ln(|w^+(t + i0)|) := \lim_{y \downarrow 0} w^+(t + iy)$ holds for a.e. $t \in \mathbb{R}$.

(ii) *If $V = V^* \in \mathfrak{S}_1(\mathfrak{H})$ and the condition*

$$(V f, f) \leq -\operatorname{Im}(H f, f), \quad f \in \operatorname{dom}(H), \quad (6.13)$$

is satisfied, then the perturbation determinant $w(z) := \det(I + iV(H - z)^{-1})$, $z \in \mathbb{C}_+$, admits the representation

$$w(z) = \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t - z} \right) \eta(t) dt \right\}, \quad z \in \mathbb{C}_+, \quad (6.14)$$

where $\eta(\cdot) \in L^1(\mathbb{R})$ is real-valued and $\eta(t) = -\ln(|w(t + i0)|)$ for a.e. $t \in \mathbb{R}$.

Proof. (i) In fact, the proof of Lemma 5.6(i) remains true if we replace B and $M(z)$ by H and z , respectively. Hence there exists a non-negative function $\eta_+(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ such that the representation (5.19) holds, i.e.

$$w_+(z) = \varkappa_+ \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) \eta_+(t) dt \right\}, \quad z \in \mathbb{C}_+, \quad (6.15)$$

where $\eta_+(t) = -\ln(|\det(w_+(t + i0))|)$ for a.e. $t \in \mathbb{R}$. It follows that

$$|w_+(z)| = \exp \left\{ -\frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(t - x)^2 + y^2} \eta_+(t) dt \right\}, \quad z = x + iy \in \mathbb{C}_+,$$

or

$$\exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(t-x)^2 + y^2} \eta_+(t) dt \right\} = \left| \frac{1}{w_+(z)} \right|, \quad z = x + iy \in \mathbb{C}_+. \quad (6.16)$$

Notice that $\frac{1}{w_+(z)} = \det(I - iV_+(H' - z)^{-1})$, where $H' := H + iV_+$. Applying the known estimate for the perturbation determinant (see [32, Section IV.1]) we arrive at the inequality

$$\frac{1}{|w_+(iy)|} \leq \exp \left\{ \|V_+(H' - iy)^{-1}\|_{\mathfrak{S}_1} \right\} \leq \exp \left\{ \frac{\|V_+\|_{\mathfrak{S}_1}}{y - \|V_+\|} \right\}, \quad y > \|V_+\|.$$

Combining this estimate with relation (6.16) this yields

$$\frac{1}{\pi} \int_{\mathbb{R}} \frac{y^2}{t^2 + y^2} \eta_+(t) dt \leq \|V_+\|_{\mathfrak{S}_1} \frac{y}{y - \|V_+\|}, \quad y > \|V_+\|.$$

In turn, tending y to $+\infty$ and applying the monotone convergence theorem yields

$$\|\eta_+\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} \eta_+(t) dt \leq \pi \|V_+\|_{\mathfrak{S}_1}.$$

Moreover, setting $\varkappa'_+ = \varkappa_+ \exp \left\{ -\frac{i}{\pi} \int_{\mathbb{R}} \frac{t}{1+t^2} \eta_+(t) dt \right\}$ and using (6.15) we arrive at the representation

$$w_+(z) = \varkappa'_+ \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \frac{1}{t-z} \eta_+(t) dt \right\}, \quad z \in \mathbb{C}_+.$$

Finally, since $\lim_{y \rightarrow \infty} w_+(iy) = 0$, we get $\varkappa'_+ = 1$ which proves (6.11).

Notice that $H - iV_+$ is a maximal dissipative operator. Obviously we have

$$w^+(z) = \det(I - iV_+(H - z)^{-1}) = \frac{1}{\det(I + iV_+(H - iV_+ - z)^{-1})}, \quad z \in \mathbb{C}_+.$$

By the result above there is a non-negative function $\eta^+(\cdot) \in L^1(\mathbb{R})$ such that the representation

$$\det(I + iV_+(H - iV_+ - z)^{-1}) = \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \frac{\eta^+(t)}{t-z} dt \right\}, \quad z \in \mathbb{C}_+,$$

holds which immediately proves (6.12).

(ii) Let $V = V_+ - V_-$, $V_{\pm} \geq 0$. We set $H_- := H - iV_-$. Notice that H_- is also a maximal accumulative operator. Setting $w_{\pm}(z) := \det(I + iV_{\pm}(H_{\pm} - z)^{-1})$, $z \in \mathbb{C}_+$ and using the chain rule (see Appendix B, formula (B.2)) we get

$$w(z) = \frac{w_+(z)}{w_-(z)}, \quad z \in \mathbb{C}_+. \quad (6.17)$$

Next, we rewrite condition (6.13) as

$$(V_+ f, f) \leq -\operatorname{Im}(H f, f) + (V_- f, f) = -\operatorname{Im}(H_- f, f), \quad f \in \operatorname{dom}(H_-).$$

Applying (i) we find that $w_+(\cdot)$ admits the representation (6.11) with $\eta_+(t) \geq 0$. Similarly, since $(V_-f, f) \leq -\text{Im}(H_-f, f)$, $f \in \text{dom}(H_-)$, we obtain by applying (i) that $w_-(\cdot)$ also admits a representation of type (6.11) with $\eta_-(t) \geq 0$ in place of $\eta_+(t)$. Combining (6.17) with these representations and setting $\eta(t) := \eta_+(t) - \eta_-(t)$, $t \in \mathbb{R}$, we arrive at (6.14). \square

The counterpart of Theorem 5.7 reads now as follows.

Theorem 6.6. *Let H be a maximal accumulative operator, $V \in \mathfrak{S}_1(\mathfrak{H})$ and let $H' = H + V$ be accumulative, i.e.*

$$\text{Im}(Vf, f) \leq -\text{Im}(Hf, f), \quad f \in \text{dom}(H). \quad (6.18)$$

Then H' is maximal accumulative and there exists a complex-valued function $\omega(\cdot) \in L^1(\mathbb{R})$ such that the following holds:

(i) *The perturbation determinant $\Delta_{H'/H}(z)$, $z \in \mathbb{C}_+$, admits the representation*

$$\Delta_{H'/H}(z) = \exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \frac{\omega(t)}{t-z} dt \right\}, \quad z \in \mathbb{C}_+. \quad (6.19)$$

(ii) *The trace formulas*

$$\text{tr}((H' - z)^{-1} - (H - z)^{-1}) = -\frac{1}{\pi} \int_{\mathbb{R}} \frac{\omega(t)}{(t-z)^2} dt, \quad z \in \mathbb{C}_+, \quad (6.20)$$

and

$$\text{tr}(V) = \frac{1}{\pi} \int_{\mathbb{R}} \omega(t) dt. \quad (6.21)$$

hold.

Proof. Clearly, H' is m -accumulative because H is so and V is bounded.

(i) Let $V = V_R + iV_I$ where $V_R := \text{Re}(V)$ and $V_I := \text{Im}(V)$. We set $K := H + V_R$ and note that K is m -accumulative. Using (6.18) we find

$$(V_I f, f) \leq -\text{Im}(Kf, f) = -\text{Im}(Hf, f), \quad f \in \text{dom}(K) = \text{dom}(H).$$

By Lemma 6.5(ii) there exists a real-valued function $\eta(\cdot) \in L^1(\mathbb{R})$ such that the perturbation determinant $\Delta_{H'/K}(z) = \det(I + iV_I(K - z)^{-1})$, $z \in \mathbb{C}_+$, admits the representation

$$\Delta_{H'/K}(z) = \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \frac{\eta(t)}{t-z} dt \right\}, \quad z \in \mathbb{C}_+. \quad (6.22)$$

Furthermore, by Lemma 6.3(ii), there exists a real-valued function $\xi(\cdot) \in L^1(\mathbb{R})$ such that the perturbation determinant $\Delta_{K/H}(z) = \det(I + V_R(H - z)^{-1})$, $z \in \mathbb{C}_+$, admits the representation

$$\Delta_{K/H}(z) = \exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \frac{\xi(t)}{t-z} dt \right\}, \quad z \in \mathbb{C}_+. \quad (6.23)$$

By the chain rule $\Delta_{H'/H}(z) = \Delta_{H'/K}(z)\Delta_{K/H}(z)$, $z \in \mathbb{C}_+$ (see formula (B.2)), and setting $\omega(t) := \xi(t) + i\eta(t)$, $t \in \mathbb{R}$, we arrive at (6.19) with a complex-valued function $\omega(\cdot) \in L^1(\mathbb{R})$.

(ii) Taking logarithmic derivative from both sides of (6.19) and using the property (B.3) we arrive at the trace formula (6.20).

Next, to prove (6.21) we rewrite (6.20) in the form

$$\operatorname{tr}((H' - z)^{-1}V(H - z)^{-1}) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\omega(t)}{(t - z)^2} dt, \quad z \in \mathbb{C}_+,$$

and put here $z = iy$. Then multiplying both sides by y^2 and tending y to $+\infty$ with account of Lemma 6.1 we arrive at (6.21). \square

Remark 6.7. Let us compare Theorem 6.6 with Krein's results of [46].

(i) Krein [46] considered the maximal accumulative operator $H' := H - iV_+$, with $H = H^*$ and $V_+ = V_+^* \geq 0$. According to [46, Theorem 9.1] there exists a non-decreasing function $\tau(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ such that the perturbation determinant $\Delta_{H/H'}(z)$, $z \in \mathbb{C}_+$, admits the representation

$$\Delta_{H/H'}(z) = \exp \left\{ i \int_{\mathbb{R}} \frac{d\tau(t)}{t - z} \right\}, \quad z \in \mathbb{C}_+. \quad (6.24)$$

Lemma 6.5(i) improves Krein's result. Indeed, it is shown that the measure $d\tau(\cdot)$ is absolutely continuous, i.e. $d\tau(t) = \eta_+(t)dt$ where $\eta_+(\cdot) \geq 0$ and $\eta_+(\cdot) \in L^1(\mathbb{R})$. Notice that in distinction to (6.19) Krein considers the perturbation determinant $\Delta_{H/H'}(z)$.

(ii) Theorem 6.6 generalizes Theorem 9.1 of [46] in two directions. Firstly, H can be m -accumulative and, secondly, condition $\operatorname{Im}(V) \leq 0$ in [46] is relaxed to (6.18).

(iii) Let $H = H^*$ and $H' = H + V$ where V is accumulative, $V \in \mathfrak{S}_1(\mathfrak{H})$ and the condition $V_I \log(-V_I) \in \mathfrak{S}_1(\mathfrak{H})$, $V_I = \operatorname{Im}(V) \leq 0$, is valid. By [2, Corollary 4.3], there exists a real-valued function $\xi(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2}dt)$ such that the trace formula

$$\operatorname{tr}((H' - z)^{-1} - (H - z)^{-1}) = -\frac{1}{\pi} \int_{\mathbb{R}} \frac{\xi(t)}{(t - z)^2} dt, \quad z \in \mathbb{C}_+, \quad (6.25)$$

is valid. Thus, alongside representation (6.20) with a complex-valued function $\omega(\cdot) \in L^1(\mathbb{R})$ there is the representation (6.25) with a real-valued function $\xi(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2}dt)$.

The trace formula (6.24) can be extended to a class of holomorphic in \mathbb{C}_- functions $\Phi(\cdot)$ admitting the representation

$$\Phi(z) = \int_{[0, \infty)} \Psi(z, t) dp(t), \quad z \in \overline{\mathbb{C}_-}, \quad (6.26)$$

where $p(\cdot)$ is a complex-valued Borel measure on $[0, \infty)$ of finite variation, i.e

$$\int_{[0, \infty)} |dp((t))| < \infty,$$

and

$$\Psi(z, t) := \begin{cases} \frac{e^{-itz} - 1}{-it}, & t > 0, \\ z, & t = 0. \end{cases}, \quad z \in \overline{\mathbb{C}_-}.$$

It is well known that any m -accumulative (in particular selfadjoint) operator H in \mathfrak{H} generates a strongly continuous semigroup of contractions e^{-itH} , $t \geq 0$. This fact allows one to define the operator $\Phi(T)$ by

$$\Phi(H)h = \int_{[0, \infty)} \Psi(H, t)h dp(t), \quad h \in \text{dom}(H). \quad (6.27)$$

In general, $\Phi(H)$ is unbounded and closable such that $\text{dom}(\Phi(H)) \supseteq \text{dom}(H)$. However, if $\text{supp}(p) \subset (0, \infty)$, then $\Phi(H)$ is bounded.

In [46, Theorem 9.2] Krein has shown that for a selfadjoint operator $H = H^*$ and a maximal accumulative operator $H' = H - iV_+$, $V_+ = V_+^* \geq 0$, the trace formula

$$\text{tr}(\Phi(H') - \Phi(H)) = -i \int_{\mathbb{R}} \Phi'(t) d\tau(t)$$

holds where $\Phi(\cdot)$ is given by (6.26) and $\tau(\cdot)$ is a non-decreasing function of finite variation such that the representation (6.24) is valid. Notice that Krein's result becomes comparable with those below if one changes the sign of the right-hand side, see Remark 6.7(i).

We generalize [46, Theorem 9.2] as follows.

Theorem 6.8. *Let the assumptions of Theorem 6.6 be satisfied and let $\Phi(\cdot)$ be a complex function in \mathbb{C}_+ of the form (6.26). Then both operators $\Phi(H')$ and $\Phi(H)$ are well defined, $\Phi(H') - \Phi(H) \in \mathfrak{S}_1(\mathfrak{H})$ and the following trace formula*

$$\text{tr}(\Phi(H') - \Phi(H)) = \frac{1}{\pi} \int_{\mathbb{R}} \Phi'(t) \omega(t) dt. \quad (6.28)$$

holds

Proof. We set

$$H_\alpha = H(I + i\alpha H)^{-1} \quad \text{and} \quad H'_\alpha = H'(I + i\alpha H')^{-1}, \quad \alpha > 0. \quad (6.29)$$

One easily verifies that H'_α and H_α are bounded accumulative operators. Moreover, it is easily seen that

$$\text{s-}\lim_{\alpha \rightarrow +0} (H'_\alpha - z)^{-1} = (H' - z)^{-1} \quad \text{and} \quad \text{s-}\lim_{\alpha \rightarrow +0} (H_\alpha - z)^{-1} = (H - z)^{-1} \quad (6.30)$$

for $z \in \mathbb{C}_+$. Further, it readily follows from definition (6.29) that

$$(H_\alpha - z)^{-1} = \frac{i\alpha}{1 - i\alpha z} I + \frac{1}{(1 - i\alpha z)^2} \left(H - \frac{z}{1 - i\alpha z} \right)^{-1}.$$

Combining this identity with a similar identity for $(H'_\alpha - z)^{-1}$ and applying (6.20) we get that for any $z \in \mathbb{C}_+$

$$\begin{aligned} \operatorname{tr} \left((H'_\alpha - z)^{-1} - (H_\alpha - z)^{-1} \right) &= -\frac{1}{(1 - i\alpha z)^2} \int_{\mathbb{R}} \frac{\omega(t)}{\left(t - \frac{z}{1 - i\alpha z} \right)^2} dt \\ &= -\int_{\mathbb{R}} \frac{\omega(t)}{\left(t - z(1 + i\alpha t) \right)^2} dt = -\frac{1}{\pi} \int_{\mathbb{R}} \frac{\omega(t)}{(1 + i\alpha t)^2} \frac{1}{\left(z - \frac{t}{1 + i\alpha t} \right)^2} dt. \end{aligned} \quad (6.31)$$

Let Γ be a simple closed curve such that its interior contains $\sigma(H'_\alpha) \cup \sigma(H_\alpha)$. Since H_α and H'_α are bounded, the Riesz-Dunford functional calculus yields

$$e^{-isH'_\alpha} = -\frac{1}{2\pi i} \oint_{\Gamma} e^{-sz} (H'_\alpha - z)^{-1} dz, \quad s \geq 0.$$

and

$$e^{-isH_\alpha} = -\frac{1}{2\pi i} \oint_{\Gamma} e^{-sz} (H_\alpha - z)^{-1} dz, \quad s \geq 0.$$

Hence

$$\begin{aligned} e^{-isH'_\alpha} - e^{-isH_\alpha} &= -\frac{1}{2\pi i} \oint_{\Gamma} e^{-isz} \left((H'_\alpha - z)^{-1} - (H_\alpha - z)^{-1} \right) dz \\ &= \frac{1}{2\pi i} \oint_{\Gamma} e^{-isz} (H'_\alpha - z)^{-1} V_\alpha (H_\alpha - z)^{-1} dz, \end{aligned} \quad (6.32)$$

where

$$\begin{aligned} V_\alpha &:= H'_\alpha - H_\alpha = H'(I + i\alpha H')^{-1} - H(I + i\alpha H)^{-1} \\ &= (I + i\alpha H')^{-1} V (I + i\alpha H)^{-1}, \quad \alpha > 0. \end{aligned} \quad (6.33)$$

Since $V \in \mathfrak{S}_1(\mathfrak{H})$, the last identity implies $V_\alpha \in \mathfrak{S}_1(\mathfrak{H})$. Combining this fact with (6.32) this yields $e^{-isH'_\alpha} - e^{-isH_\alpha} \in \mathfrak{S}_1(\mathfrak{H})$. Moreover, we get from (6.32)

$$\operatorname{tr} \left(e^{-isH'_\alpha} - e^{-isH_\alpha} \right) = -\frac{1}{2\pi i} \oint_{\Gamma} e^{-isz} \operatorname{tr} \left((H'_\alpha - z)^{-1} - (H_\alpha - z)^{-1} \right) dz.$$

Combining this formula with (6.31) we obtain

$$\begin{aligned} \operatorname{tr} \left(e^{-isH'_\alpha} - e^{-isH_\alpha} \right) &= \\ \frac{1}{\pi} \int_{\mathbb{R}} dt \frac{\omega(t)}{(1 + i\alpha t)^2} \frac{1}{2\pi i} \oint_{\Gamma} \frac{e^{-isz}}{\left(z - \frac{t}{1 + i\alpha t} \right)^2} dz &= \frac{-is}{\pi} \int_{\mathbb{R}} e^{-is \frac{t}{1 + i\alpha t}} \frac{\omega(t)}{(1 + i\alpha t)^2} dt \end{aligned} \quad (6.34)$$

Further, it follows from [37, formula (IX.2.22)] and (6.33) that for $s > 0$

$$\begin{aligned} e^{-isH'_\alpha} - e^{-isH_\alpha} = \\ -i \int_0^s e^{-i(s-x)H'_\alpha} (I + i\alpha H')^{-1} V (I + i\alpha H)^{-1} e^{-ixH_\alpha} dx, \end{aligned} \quad (6.35)$$

and

$$e^{-isH'} - e^{-isH} = -i \int_0^s e^{-i(s-x)H'} V e^{-ixH} dx, \quad s > 0. \quad (6.36)$$

Since $V \in \mathfrak{S}_1(\mathfrak{H})$ we find $e^{-isH'_\alpha} - e^{-isH_\alpha} \in \mathfrak{S}_1(\mathfrak{H})$ and $e^{-isH'} - e^{-isH} \in \mathfrak{S}_1(\mathfrak{H})$, $s > 0$, see above. Moreover, they imply the following important estimates

$$\left\| e^{-isH'_\alpha} - e^{-isH_\alpha} \right\|_{\mathfrak{S}_1} \leq s \|V_\alpha\|_{\mathfrak{S}_1} \text{ and } \left\| e^{-isH'} - e^{-isH} \right\|_{\mathfrak{S}_1} \leq s \|V\|_{\mathfrak{S}_1}. \quad (6.37)$$

Since $V \in \mathfrak{S}_1(\mathfrak{H})$, it follows from (6.30) and (6.33) that $\lim_{\alpha \rightarrow 0} \|V_\alpha - V\|_{\mathfrak{S}_1} = 0$. Combining this relation with integral representations (6.35) and (6.36) we obtain

$$\lim_{\alpha \rightarrow +0} \text{tr} \left(e^{-isH'_\alpha} - e^{-isH_\alpha} \right) = \text{tr} \left(e^{-isH'} - e^{-isH} \right), \quad s > 0. \quad (6.38)$$

Since $\omega(\cdot) \in L^1(\mathbb{R})$ and $\alpha > 0$ the dominated convergence theorem (with majorizing function $|\omega|$) yields

$$\lim_{\alpha \rightarrow +0} \frac{-is}{\pi} \int_{\mathbb{R}} e^{-is \frac{t}{1+i\alpha t}} \frac{\omega(t)}{(1+i\alpha t)^2} dt = \frac{-is}{\pi} \int_{\mathbb{R}} e^{-ist} \omega(t) dt, \quad s > 0. \quad (6.39)$$

Taking into account (6.34), (6.38) and (6.39) we obtain

$$\text{tr} \left(e^{-isH'} - e^{-isH} \right) = \frac{-is}{\pi} \int_{\mathbb{R}} e^{-ist} \omega(t) dt, \quad s > 0. \quad (6.40)$$

On the other hand, since both H and H' are m -accumulative (6.27) yields the representation

$$\Phi(H') - \Phi(H) = \int_{[0, \infty)} \frac{e^{-itH'} - e^{-itH}}{-it} dp(t). \quad (6.41)$$

Since the measure p is finite, we obtain from (6.41) and (6.37) that $\Phi(H') - \Phi(H) \in \mathfrak{S}_1(\mathfrak{H})$ and

$$\|\Phi(H') - \Phi(H)\|_{\mathfrak{S}_1} \leq \|V\|_{\mathfrak{S}_1} \int_{[0, \infty)} |dp(t)|.$$

Combining (6.40) with (6.41) we finally obtain

$$\begin{aligned} \text{tr} (\Phi(H') - \Phi(H)) &= \int_{[0, \infty)} \frac{\text{tr} (e^{-itH'} - e^{-itH})}{-it} dp(t) \\ &= \frac{1}{\pi} \int_{[0, \infty)} dp(t) \int_{\mathbb{R}} e^{-itx} \omega(x) dx = \frac{1}{\pi} \int_{\mathbb{R}} dx \omega(x) \int_{[0, \infty)} e^{-itx} dp(t). \end{aligned}$$

By $\Phi'(x) = \int_{[0, \infty)} e^{-itx} dp(t)$, $x \in \mathbb{R}$, we complete the proof. \square

In fact the class of functions Φ introduced above is not optimal even for selfadjoint operators. A more optimal class of functions in case of selfadjoint operators was introduced in [5, 54].

6.2 Pairs $\{H, H'\}$ with one m -accumulative operator

Our next goal is to prove trace formulas for pairs $\{H, H'\}$ with m -accumulative operator H , i.e. to remove the condition (6.18). For this purpose we need an analog of Lemma 5.12. To this end we recall a simple statement on the behavior at infinity of a Blaschke products $\mathcal{B}(\cdot)$ with non-real zeros $\Lambda := \{\lambda_j\}_{j \in \mathbb{N}}$ satisfying an additional assumption

$$\sum_{j=1}^{\infty} |\operatorname{Im} \lambda_j| < \infty \quad (6.42)$$

Lemma 6.9 ([46, Lemma 8.1]). *Let $\Lambda := \{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{C} \setminus \mathbb{R}$. If the condition (6.42) is satisfied, then*

$$\lim_{y \uparrow \infty} y^2 \sum_{j=1}^{\infty} \frac{\operatorname{Im}(\lambda_j)}{(iy - \lambda_j)(iy - \bar{\lambda}_j)} = - \sum_{j=1}^{\infty} \operatorname{Im} \lambda_j.$$

and the (regularized) Blaschke product

$$\tilde{\mathcal{B}}(z) = \prod_{j=1}^{\infty} \frac{z - \lambda_j}{z - \bar{\lambda}_j}$$

converges uniformly on any compact subset $\mathcal{K} \subset \mathbb{C}$ satisfying $\operatorname{dist}(\mathcal{K}, \Lambda) > 0$. Moreover, the following relations hold

$$\lim_{y \uparrow \infty} \tilde{\mathcal{B}}(z) = 1 \quad \text{and} \quad \lim_{y \uparrow \infty} y \ln |\tilde{\mathcal{B}}(z)| = -2 \sum_{j=1}^{\infty} \operatorname{Im} \lambda_j, \quad z = x + iy.$$

Now we are ready to state a counterpart of Lemma 5.12 for additive perturbations.

Lemma 6.10. *Let H be a maximal accumulative operator in \mathfrak{H} .*

(i) *If $0 \leq V_+ = V_+^* \in \mathfrak{S}_1(\mathfrak{H})$ and the condition*

$$(V_+ f, f) \leq -2 \operatorname{Im}(H f, f), \quad f \in \operatorname{dom}(H),$$

is satisfied, then the function $w_+(z) := \det(I + iV_+(H - z)^{-1})$, $z \in \mathbb{C}_+$, admits the representation

$$w_+(z) = \prod_{j=1}^{\infty} \left(\frac{z - z_j^+}{z - \bar{z}_j^+} \right)^{m_j^+} \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \frac{1}{t - z} d\mu_+(t) \right\}, \quad z \in \mathbb{C}_+, \quad (6.43)$$

where $\mu_+(\cdot)$ is a non-negative finite Borel measure, $\{z_j^+\}_{j=1}^{\infty}$ is the set of zeros of $w_+(\cdot)$ in \mathbb{C}_+ , and $\{m_j^+\}_{j=1}^{\infty}$ is the sequence of the corresponding multiplicities.

(ii) If $V = V^* \in \mathfrak{S}_1(\mathfrak{H})$ and the condition

$$(Vf, f) \leq -2\operatorname{Im}(Hf, f), \quad f \in \operatorname{dom}(H), \quad (6.44)$$

is satisfied, then the function $w(z) = \det(I + V(H - z)^{-1})$, $z \in \mathbb{C}_+$, admits the representation (6.43) where the measure $\mu_+(\cdot)$ is replaced by a real-valued measure $\mu(\cdot)$ satisfying $\int_{\mathbb{R}} |d\mu(t)| < \infty$, $\{z_j^+\}_{j=1}^\infty \subset \mathbb{C}_+$ are the zeros of the function $w(z)$ in \mathbb{C}_+ and $\{m_j^+\}_{j=1}^\infty$ their corresponding multiplicities.

Proof. (i) We set $H' = H + iV_+$. Following the proof of Lemma 5.12 with $M(z)$ replaced by z , we arrive at the representation (5.52). Obviously, we have

$$|w_+(z)| = |\mathcal{B}_+(z)| \exp \left\{ -\frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(t-x)^2 + y^2} d\mu_+(t) \right\}, \quad z = x + iy \in \mathbb{C}_+,$$

which yields

$$\exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(t-x)^2 + y^2} d\mu_+(t) \right\} = |\mathcal{B}_+(z)| \left| \frac{1}{w_+(z)} \right|, \quad z \in \mathbb{C}_+ \setminus \bigcup_{j=1}^\infty \{z_j^+\},$$

where $\sigma(H') \cap \mathbb{C}_+ = \bigcup_{j=1}^\infty \{z_j^+\}$. One easily gets

$$\frac{1}{w_+(z)} = \det \left(I - i\sqrt{V_+}(H' - z)^{-1}\sqrt{V_+} \right), \quad z \in \mathbb{C}_+ \setminus \sigma(H').$$

Combining this identity with the previous one and noting that $|\tilde{\mathcal{B}}(z)| \leq 1$, $z \in \mathbb{C}_+$, we obtain

$$\exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(t-x)^2 + y^2} d\mu_+(t) \right\} \leq \left| \det \left(I - i\sqrt{V_+}(H' - z)^{-1}\sqrt{V_+} \right) \right|,$$

$z \in \mathbb{C}_+ \setminus \sigma(H')$. In turn, combining this inequality with a simple known estimate (see [32, Section IV.1]) we arrive at the estimate

$$\exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(t-x)^2 + y^2} d\mu_+(t) \right\} \leq \exp \left\{ \|V_+(H' - z)^{-1}\|_{\mathfrak{S}_1} \right\}, \quad z \in \mathbb{C}_+ \setminus \sigma(H'),$$

which is equivalent to

$$\frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(t-x)^2 + y^2} d\mu_+(t) \leq \|V_+(H' - z)^{-1}\|_{\mathfrak{S}_1}, \quad z = x + iy \in \mathbb{C}_+ \setminus \sigma(H').$$

Further, combining this estimate with the following one

$$\|V_+(H' - z)^{-1}\|_{\mathfrak{S}_1} \leq \|V_+\|_{\mathfrak{S}_1} \frac{1}{y - \|V_+\|}, \quad y > \|V_+\|,$$

we obtain

$$\frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(t-x)^2 + y^2} d\mu_+(t) \leq \|V_+\|_{\mathfrak{S}_1} \frac{1}{y - \|V_+\|}, \quad y > \|V_+\|.$$

Multiplying both sides by y and tending y to infinity we derive

$$\int_{\mathbb{R}} d\mu_+(t) \leq \|V_+\|_{\mathfrak{S}_1} = \text{tr}(V_+).$$

The zeros z_j^+ of $w_+(z)$ lying in C_+ and their multiplicities m_j^+ coincide with the eigenvalues of $H' := H + V_+$ and their algebraic multiplicities. Since $\text{Im}(Hf, f) \leq 0$, $f \in \text{dom}(H)$, we have

$$\text{Im}(H'f, f) = \text{Im}(Hf, f) + (V_+f, f) \leq (V_+f, f), \quad f \in \text{dom}(H), \quad (6.45)$$

Denoting by \mathfrak{H}_p^+ the (closed) invariant subspace of H' spanned by the (finite-dimensional) root subspaces $\mathfrak{L}_{z_j^+} := \ker(H' - z_j^+)^{m_j}$, $j \in \mathbb{N}$, and choose a Schur orthonormal basis $\{f_k\}_{k \in \mathbb{N}}$ in \mathfrak{H}_p^+ such that in this basis the matrix of the operator $H' \upharpoonright \mathfrak{H}_p^+$ is triangular. Taking into account (6.45) we get

$$\begin{aligned} 0 &\leq \sum_j m_j^+ \text{Im}(\lambda_j^+) = \sum_k \text{Im}(H'f_k, f_k) \\ &= \sum_k \text{Im}(Hf_k, f_k) + \sum_k (V_+f_k, f_k) \leq \text{tr}(V_+) < \infty. \end{aligned}$$

By Lemma 6.9 the product $\tilde{B}_+(z) = \prod_k \left(\frac{z - z_j^+}{z - z_j^+} \right)^{m_j^+}$ converges uniformly on compact subsets of \mathbb{C}_+ . It is easily seen that $\mathcal{B}_+(z) = \tilde{\kappa} \tilde{B}_+(z)$ where $|\tilde{\kappa}| = 1$. Again by Lemma 6.9 we have $\lim_{y \uparrow \infty} \tilde{B}_+(iy) = 1$. It follows from (5.52) that

$$w_+(z) = \varkappa' \tilde{B}(z) \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \frac{1}{t - z} d\mu_+(t) \right\}, \quad z \in \mathbb{C}_+, \quad (6.46)$$

where

$$\varkappa' = \varkappa \tilde{\varkappa} \exp \left\{ i \left(\alpha_+ - \frac{1}{\pi} \int_{\mathbb{R}} \frac{t}{1 + t^2} d\mu_+(t) \right) \right\}$$

Since $\lim_{y \uparrow \infty} w_+(iy) = 1$ we immediately obtain $\varkappa' = 1$ which proves (6.43).

(ii) We set $K := H - iV_-$ where $V := V_+ - V_-$, $V_{\pm} \geq 0$. We note K is maximal accumulative. Using the chain rule for perturbation determinants (see formula (B.2)) we easily get

$$w(z) = \frac{\det(I + iV_+(K - z)^{-1})}{\det(I + iV_-(K - z)^{-1})} =: \frac{w_+(z)}{w_-(z)}, \quad z \in \mathbb{C}_+. \quad (6.47)$$

Rewriting (6.44) in the form

$$(V_+f, f) - (V_-f, f) \leq -2\text{Im}(Hf, f), \quad f \in \text{dom}(H),$$

we get for any $f \in \text{dom}(K) = \text{dom}(H)$

$$(V_+f, f) \leq -2\text{Im}(Hf, f) + (V_-f, f) \leq -2\text{Im}(Hf, f) + 2(V_-f, f) = -\text{Im}(Kf, f).$$

According to statement (i) the perturbation determinant $w_+(z) := \det(I + iV_+(K - z)^{-1})$, admits the representation (6.43) $z \in \mathbb{C}_+$. Since

$$(V_-f, f) \leq -\operatorname{Im}(Hf, f) + (V_-f, f) = -\operatorname{Im}(Kf, f), \quad f \in \operatorname{dom}(K) = \operatorname{dom}(H),$$

Lemma 6.5(i) yields the following representation

$$\det(I + iV_-(K - z)^{-1}) = \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \frac{\eta_+(t)}{t - z} dt \right\}, \quad z \in \mathbb{C}_+. \quad (6.48)$$

Inserting (6.43) and (6.48) into (6.47) we arrive at the representation

$$w(z) = \prod_{j=1}^{\infty} \left(\frac{z - z_j^+}{z - \overline{z_j^+}} \right)^{m_j^+} \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \frac{1}{t - z} d\mu(t) \right\}, \quad z \in \mathbb{C}_+.$$

Here $d\mu(t) = d\mu_+(t) - \eta_+ dt$, $\{z_j^+\}_{j=1}^{\infty}$ is the set zeros of $w_+(\cdot)$ lying in \mathbb{C}_+ and $\{m_j^+\}_{j=1}^{\infty}$ the set of the corresponding multiplicities. Since the operator H is m -accumulative, the function $w_-(z) = \det(I + iV_-(K - z)^{-1}) = \Delta_{H/K}(z)$ has no zeros in \mathbb{C}_+ . Combining this fact with representation (6.43) for $w_+(\cdot)$ we conclude that the set $\{z_j^+\}_{j=1}^{\infty}$ is the set of zeros of $w(\cdot)$ in \mathbb{C}_+ with the corresponding multiplicities $\{m_j^+\}_{j=1}^{\infty}$. \square

Now we are ready to obtain the trace formulas for a pair $\{H, H + V\}$ with m -accumulative operator H . A counterpart of Theorem 5.13 takes the following form for additive perturbations.

Theorem 6.11. *Let H be a maximal accumulative operator in \mathfrak{H} , $V \in \mathfrak{S}_1(\mathfrak{H})$ and let $H' := H + V$. Then the following holds:*

(i) *There exists a complex-valued Borel measure $d\nu(t) := id\mu_+(t) + \omega(t)dt$ on \mathbb{R} such that $d\mu_+(\cdot)$ is a non-negative finite Borel measure on \mathbb{R} , $\omega(\cdot) \in L^1(\mathbb{R})$ and the perturbation determinant $\Delta_{H'/H}(\cdot)$ admits the representation*

$$\Delta_{H'/H}(z) = \prod_{j=1}^{\infty} \left(\frac{z - z_j^+}{z - \overline{z_j^+}} \right)^{m_j^+} \exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{t - z} d\nu(t) \right\}, \quad z \in \mathbb{C}_+. \quad (6.49)$$

where $d\nu(t) := id\mu_+(t) + \omega(t)dt$, $\{z_j^+\}_{j=1}^{\infty}$ is the set of eigenvalues of H' in \mathbb{C}_+ , and $\{m_j^+\}_{j=1}^{\infty}$ is the set of corresponding algebraic multiplicities.

(ii) *The trace formula holds*

$$\begin{aligned} \operatorname{tr} \left((H' - z)^{-1} - (H - z)^{-1} \right) = \\ - \sum_{j=1}^{\infty} \frac{2i m_j^+ \cdot \operatorname{Im}(z_j^+)}{(z - z_j^+)(z - \overline{z_j^+})} - \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\nu(t)}{(t - z)^2}, \quad z \in \mathbb{C}_+. \end{aligned} \quad (6.50)$$

In particular, one has

$$\begin{aligned}\operatorname{tr}(V) &= 2i \sum_{j=1}^{\infty} m_j^+ \operatorname{Im}(z_j^+) + \frac{1}{\pi} \int_{\mathbb{R}} d\nu(t) \\ &= 2i \sum_j m_j^+ \operatorname{Im}(z_j^+) + \frac{i}{\pi} \int_{\mathbb{R}} d\mu_+(t) + \frac{1}{\pi} \int_{\mathbb{R}} \omega(t) dt.\end{aligned}\tag{6.51}$$

and

$$\operatorname{tr}(V_I) = 2 \sum_j m_j^+ \operatorname{Im}(z_j^+) + \frac{1}{\pi} \int_{\mathbb{R}} d\mu_+(t) + \frac{1}{\pi} \int_{\mathbb{R}} \omega_I(t) dt.\tag{6.52}$$

where $V_I := \operatorname{Im}(V)$ and $\omega_I(t) := \operatorname{Im}(\omega(t)) \leq 0$, $t \in \mathbb{R}$.

Proof. (i) Let $V_R := \operatorname{Re}(V)$ and $V_I = \operatorname{Im}(V)$. Further, let $V_I = V_I^+ - V_I^-$ be the spectral decomposition of V_I , i.e. $V_I^{\pm} \geq 0$ and $V_I^+ V_I^- = V_I^- V_I^+ = 0$. We set $K := H + \tilde{V}$ and $\tilde{V} := V_R - i|V_I|$, where $|V_I| = V_I^+ + V_I^-$. Clearly, the operator K is m -accumulative because so are H and \tilde{V} ($\in [\mathfrak{H}]$).

We put $w_+(z) := \Delta_{H'/K}(z) = \det(I + 2V_I^+(K - z)^{-1})$, $z \in \mathbb{C}_+$. It is easily seen that $(2V_I^+ f, f) \leq -2\operatorname{Im}(Kf, f)$, $f \in \operatorname{dom}(K)$. Therefore applying Lemma 6.10(i) we arrive at the representation (6.43) for $w_+(\cdot)$ with the *non-negative finite Borel measure* $d\mu_+(\cdot)$, the set $\{z_j^+\}_{j=1}^{\infty}$ of zeros of $w_+(\cdot)$ lying in \mathbb{C}_+ and the set $\{m_j^+\}_{j=1}^{\infty}$ of corresponding multiplicities. However, the zeros $\{z_j^+\}_{j=1}^{\infty}$ and their multiplicities $\{m_j^+\}_{j=1}^{\infty}$ coincide with the eigenvalues of H' lying in \mathbb{C}_+ and their algebraic multiplicities, respectively (see Appendix B, property 4).

Further, since H is accumulative, we have

$$\operatorname{Im}(\tilde{V}f, f) = -(|V|f, f) \leq -\operatorname{Im}(Hf, f), \quad f \in \operatorname{dom}(H).$$

Therefore, by Theorem 6.6(i), there exists a *complex-valued function* $\omega(\cdot) \in L^1(\mathbb{R})$ such that the following representation holds

$$\Delta_{K/H}(z) = \det(I + (V_R - i|V_I|)(H - z)^{-1}) = \exp \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \frac{\omega(t)}{t - z} dt \right\}, \tag{6.53}$$

$z \in \mathbb{C}_+$. Setting $d\nu(t) := id\mu_+(t) + \omega(t)dt$ we define a complex-valued Borel measure on \mathbb{R} satisfying $\int_{\mathbb{R}} |d\nu(t)| < \infty$. Finally, combining representation (6.43) for $w_+(\cdot) := \Delta_{H'/K}(\cdot)$ with representation (6.53) for $\Delta_{K/H}(\cdot)$ and using the identity $\Delta_{H'/H}(\cdot) = \Delta_{H'/K}(\cdot) \Delta_{K/H}(\cdot)$ (see (B.2)), we arrive at representation (6.49).

(ii) Clearly, $\{z \in \mathbb{C} : \operatorname{Im}(z) > \|V_I^+\|\} \subset \rho(H')$. Therefore formula (B.3) can be applied to the determinant $\Delta_{H'/H}(z)$ for $\operatorname{Im}(z) > \|V_I^+\|$. Taking the logarithmic derivative of both sides of (6.43) and applying (B.3) we obtain

$$\operatorname{tr}((H - z)^{-1} - (H' - z)^{-1}) = \sum_j m_j^+ \left(\frac{1}{z - z_j^+} - \frac{1}{\overline{z - z_j^+}} \right) + \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\nu(t)}{(t - z)^2},$$

for $\text{Im}(z) > \|V_I^+\|$ which proves (6.50). Since V is bounded one rewrites this identity as

$$\text{tr}((H' - z)^{-1}V(H - z)^{-1}) = \sum_j \frac{2i m_j^+ \cdot \text{Im}(z_j^+)}{(z - z_j^+)(z - \overline{z_j^+})} + \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\nu(t)}{(t - z)^2}.$$

Setting here $z = iy$, $y > \|V_I^+\|$, multiplying both sides by y^2 and passing to the limit as $y \uparrow \infty$ we obtain by combining Lemma 6.1 with Lemma 6.9 and applying the dominated convergence theorem the relation

$$\begin{aligned} -\text{tr}(V) &= -2i \sum_j m_j^+ \text{Im}(z_j^+) + \lim_{y \uparrow \infty} \frac{y^2}{\pi} \int_{\mathbb{R}} \frac{d\nu(t)}{(t - iy)^2} \\ &= -2i \sum_j m_j^+ \text{Im}(z_j^+) - \frac{1}{\pi} \int_{\mathbb{R}} d\nu(t) \\ &= -2i \sum_j m_j^+ \text{Im}(z_j^+) - \frac{i}{\pi} \int_{\mathbb{R}} d\mu_+(t) - \frac{1}{\pi} \int_{\mathbb{R}} \omega(t) dt. \end{aligned}$$

follows which implies (6.51). In turn, (6.51) yields (6.52). \square

7 Examples

7.1 Matrix Sturm-Liouville operators on \mathbb{R}_+

Let us consider the matrix Sturm-Liouville differential expression in $L^2(\mathbb{R}_+, \mathbb{C}^n)$

$$(\mathcal{A}f)(x) := -\frac{d^2}{dx^2}f(x) + Q(x)f(x), \quad f = \text{col}\{f_1, \dots, f_n\}, \quad (7.1)$$

with $n \times n$ selfadjoint matrix potential $Q(\cdot) = Q(\cdot)^* \in L_{\text{loc}}^1(\mathbb{R}_+, \mathbb{C}^{n \times n})$.

Denote by $A = A_{\min}$ and A_{\max} the minimal and the maximal operators, respectively associated on $L^2(\mathbb{R}_+, \mathbb{C}^n)$ with the differential expression (7.1). Clearly, A is symmetric. Assume also that \mathcal{A} is limit point at infinity, i.e. the deficiency indices are minimal, $n_{\pm}(A) = n$. It is known (see, for instance, [52, Section 5.17.4]) that $A^* = A_{\max}$. The latter means that the domain $\text{dom}(A^*)$ is locally regular, i.e.

$$\text{dom}(A^*) \subset W_{\text{loc}}^{2,2}(\mathbb{R}_+, \mathbb{C}^n) \quad \text{and} \quad \chi_{[0,b]} \text{dom}(A^*) = W^{2,2}([0,b], \mathbb{C}^n) \quad (7.2)$$

for any $b > 0$, where $\chi_{\delta}(\cdot)$ stands for the indicator of a Borel subset δ . A^* is given by the differential expression (7.1) on the domain $\text{dom}(A^*)$. Therefore the trace operators $\Gamma_0, \Gamma_1 : \text{dom}(A^*) \rightarrow \mathbb{C}^n$,

$$\Gamma_0 f = f(0), \quad \Gamma_1 f = f'(0), \quad f = \text{col}\{f_1, \dots, f_n\},$$

are well defined and the Green identity (2.1) holds. Moreover, one easily proves that $\Pi = \{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ forms a boundary triplet for A^* . Hence the minimal operator $A = A_{\min}$ is a restriction of A^* to the domain

$$\text{dom}(A) = \ker \Gamma_0 \cap \ker \Gamma_1 = \{f \in \text{dom}(A^*) : f'(0) = f(0) = 0\},$$

and due to (7.2) the regularity property $\text{dom}(A) \subset W_{0,\text{loc}}^{2,2}(\mathbb{R}_+, \mathbb{C}^n)$ holds.

Notice that $\text{dom}(A^*) = W^{2,2}(\mathbb{R}, \mathbb{C}^n)$ whenever $Q \in L^\infty(\mathbb{R}_+, \mathbb{C}^{n \times n})$. Let

$$\Psi_j(z, x) = \text{col} \{ \Psi_{j_1}(z, x), \dots, \Psi_{j_n}(z, x) \}, \quad j \in \{1, \dots, n\},$$

be a basis in $\mathfrak{N}_z(A) = \ker(A^* - z)$ and let $\Psi(z, x) := (\Psi_1(z, x), \dots, \Psi_n(z, x)) = (\Psi_{kj}(z, x))_{k,j=1}^n$ be the Weyl $n \times n$ -matrix solution of the equation $A^* f = z f$. Then the corresponding Weyl function is

$$M(z) = \Psi'(z, 0) \Psi(z, 0)^{-1}.$$

We note that the assumption $n_\pm(A) = n$ is satisfied whenever $Q(\cdot) = Q(\cdot)^* \in L^1(\mathbb{R}_+, \mathbb{C}^{n \times n}) \cap L^\infty(\mathbb{R}_+, \mathbb{C}^{n \times n})$. In this case the Weyl matrix solution $\Psi(z, x)$ is proportional to the so-called Jost function $F(z, \cdot)$ which solves the equation

$$F(z, x) = e^{i\sqrt{z}x} I_n - \int_x^\infty \frac{1}{\sqrt{z}} \sin(\sqrt{z}(x-t)) Q(t) F(z, t) dt, \quad z \in \mathbb{C} \setminus \{0\}, \quad (7.3)$$

where $\text{Im}(\sqrt{z}) \geq 0$. Clearly, $\Psi(z, x) = F(z, x) C(z)$ where $C(\cdot)$ is a non-singular $n \times n$ -matrix function, $\det C(z) \neq 0$, $z \in \mathbb{C}_\pm$. In this case the Weyl function is $M(z) = F'(z, 0) F(z, 0)^{-1}$.

Let \tilde{A}' and \tilde{A} be proper extensions of A . Since $n_\pm(A) = n < \infty$ condition (1.6) is always satisfied. If both \tilde{A}' and \tilde{A} are disjoint from $A_0 := A^* \upharpoonright \ker(\Gamma_0)$, then, by Proposition 2.3, there exist bounded operators $B', B \in [\mathbb{C}^n]$ such that $\tilde{A}' = A_{B'}$ and $\tilde{A} = A_B$, i.e. $\text{dom}(\tilde{A}') = \{f \in \text{dom}(A^*) : f'(0) = B' f(0)\}$ and $\text{dom}(\tilde{A}) = \{f \in \text{dom}(A^*) : f'(0) = B f(0)\}$. Thus, the boundary triplet Π is regular for the pair $\{\tilde{A}', \tilde{A}\}$ which yields $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$. By Lemma 4.1(iv) we get

$$\Delta_{\tilde{A}'/\tilde{A}}^\Pi(z) = \frac{\det(B' - M(z))}{\det(B - M(z))} = \frac{\det(B' F(z, 0) - F'(z, 0))}{\det(B F(z, 0) - F'(z, 0))}, \quad z \in \rho(\tilde{A}) \cap \rho(A_0).$$

In particular, if $Q \equiv 0$, then $M_0(z) = i\sqrt{z}I_n$ and

$$\Delta_{\tilde{A}'/\tilde{A}}^\Pi(z) = \frac{\det(B' - M_0(z))}{\det(B - M_0(z))} = \frac{\det(B' - i\sqrt{z})}{\det(B - i\sqrt{z})}, \quad z \in \rho(\tilde{A}) \cap \rho(A_0).$$

If $\tilde{A} = A_0$, then the boundary triplet Π is not regular for $\{\tilde{A}', \tilde{A}\}$. Nevertheless, by Corollary 4.4(i), there exists a boundary triplet $\tilde{\Pi} = \{\mathcal{H}, \tilde{\Gamma}_0, \tilde{\Gamma}_1\}$ for A^* such that $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^{\tilde{\Pi}}$. If \tilde{A}' is disjoint from A_1 , then $0 \in \rho(B')$. Thus, by Proposition 4.9(iii), there exists $\mu = 0$ and a constant $c \in \mathbb{C}$ such that

$$\Delta_{\tilde{A}', \tilde{A}}^{\tilde{\Pi}}(z) = c \det(I_n - (B')^{-1} M(z)) = \frac{c}{\det(B')} \frac{\det(B' F(z, 0) - F'(z, 0))}{\det(F(z, 0))},$$

for $z \in \rho(\tilde{A})$. If $\tilde{A}' = A_1$, then $B' = 0$ and applying Proposition 4.9(iii) with $\mu = 1$ we find a constant $c \in \mathbb{C}$ such that

$$\Delta_{\tilde{A}', \tilde{A}}^{\tilde{\Pi}}(z) = c \det(M(z)) = c \frac{\det(F'(z, 0))}{\det(F(z, 0))}, \quad z \in \rho(\tilde{A}). \quad (7.4)$$

7.2 Sturm-Liouville operators on $(0, b)$

Next we consider the differential expression (7.1) in $L^2([0, b], \mathbb{C}^n)$ with $n \times n$ -matrix potential $Q(\cdot) = Q(\cdot)^* \in L^2([0, b], \mathbb{C}^{n \times n})$. It is well known that the maximal operator A_{\max} associated in $L^2([0, b], \mathbb{C}^n)$ with the differential expression \mathcal{A}

$$(\mathcal{A}[f])(x) := -\frac{d^2}{dx^2}f(x) + Q(x)f(x)$$

is given by

$$(A^*f)(x) := (\mathcal{A}[f])(x), \quad f \in \text{dom}(A^*) = W^{2,2}((0, b), \mathbb{C}^n). \quad (7.5)$$

where $f = \text{col}\{f_1, \dots, f_n\}$. The minimal operator $A = A_{\min}$ is a closed symmetric operator given by

$$\begin{aligned} (Af)(x) &:= (\mathcal{A}[f])(x), \\ f \in \text{dom}(A) &:= \left\{ W^{2,2}((0, b), \mathbb{C}^n) : \begin{array}{l} f(0) = f'(0) = 0 \\ f(b) = f'(b) = 0 \end{array} \right\} \end{aligned} \quad (7.6)$$

Notice that $A_{\max} = A^*$. Due to the regularity property $\text{dom}(A^*) = W^{2,2}((0, b), \mathbb{C}^n)$ the mappings

$$\Gamma_0 f := \begin{pmatrix} f(b) \\ -f(0) \end{pmatrix}, \quad \Gamma_1 f := \begin{pmatrix} -f'(b) \\ -f'(0) \end{pmatrix}, \quad f \in W^{2,2}((0, b), \mathbb{C}^n), \quad (7.7)$$

are well defined. Moreover, one easily checks that $\Pi = \{\mathbb{C}^{2n}, \Gamma_0, \Gamma_1\}$ forms a boundary triplet for A^* . Notice that $A_0 := A^* \upharpoonright \ker(\Gamma_0)$ and $A_1 := A^* \upharpoonright \ker(\Gamma_1)$ correspond to the Dirichlet and Neumann extensions, respectively.

Let us introduce the $n \times n$ matrix solutions $C(z, x)$ and $S(z, x)$

$$\begin{aligned} \mathcal{A}[C(z, x)] &= zC(z, x), \quad C(z, 0) = I_n, \quad C'(z, 0) = 0_n \\ \mathcal{A}[S(z, x)] &= zS(z, x), \quad S(z, 0) = 0_n, \quad S'(z, 0) = I_n. \end{aligned}$$

Any $f_z \in \ker(A^* - z)$ admits the representation $f_z(x) = C(z, x)\xi + S(z, x)\eta$ for some $\xi, \eta \in \mathbb{C}^n$. Hence

$$\Gamma_0 f_z = \begin{pmatrix} C(z, b) & S(z, b) \\ -I_n & 0_n \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad \text{and} \quad \Gamma_1 f_z = \begin{pmatrix} -C'(z, b) & -S'(z, b) \\ 0_n & -I_n \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

Combining these relations with Definition 2.4 we find that the Weyl function $M(\cdot)$ corresponding to the triplet Π is

$$\begin{aligned} M(z) &= \begin{pmatrix} -C'(z, b) & -S'(z, b) \\ 0_n & -I_n \end{pmatrix} \begin{pmatrix} 0_n & -I_n \\ S(z, b)^{-1} & S(z, b)^{-1}C(z, b) \end{pmatrix} \\ &= \begin{pmatrix} -S'(z, b)S(z, b)^{-1} & C'(z, b) - S'(z, b)S(z, b)^{-1}C(z, b) \\ -S(z, b)^{-1} & -S(z, b)^{-1}C(z, b) \end{pmatrix} \\ &= - \begin{pmatrix} S'(z, b)S(z, b)^{-1} & S^*(z, b)^{-1} \\ S(z, b)^{-1} & -S(z, b)^{-1}C(z, b) \end{pmatrix}, \quad z \in \mathbb{C}_{\pm}. \end{aligned} \quad (7.8)$$

If $Q \equiv 0$, then the Weyl function $M_0(z)$ is

$$M_0(z) = -\frac{1}{\sin(\sqrt{z}b)} \begin{pmatrix} \sqrt{z} \cos(\sqrt{z}b)I_n & I_n \\ I_n & -\cos(\sqrt{z}b)I_n \end{pmatrix}. \quad (7.9)$$

Let $B' = \begin{pmatrix} B'_{11} & B'_{12} \\ B'_{21} & B'_{22} \end{pmatrix}$ and $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ where $B'_{ij}, B_{ij} \in \mathbb{C}^{n \times n}$, $i, j \in \{1, 2\}$. Due to (7.7) the extensions $\tilde{A}' = A_{B'}$ and $\tilde{A} = A_B$ are given by

$$\begin{aligned} \tilde{A}' &= (\mathcal{A}[f])(x), \quad f \in \text{dom}(\tilde{A}'), \\ \text{dom}(\tilde{A}') &= \left\{ f \in W^{2,2}((0, b), \mathbb{C}^n) : \begin{aligned} f'(b) &= -B'_{11}f(b) + B'_{12}f(0) \\ f'(0) &= -B'_{21}f(b) + B'_{22}f(0) \end{aligned} \right\} \end{aligned}$$

and

$$\begin{aligned} \tilde{A} &= (\mathcal{A}[f])(x), \quad f \in \text{dom}(\tilde{A}), \\ \text{dom}(\tilde{A}) &= \left\{ f \in W^{2,2}(0, b) : \begin{aligned} f'(b) &= -B_{11}f(b) + B_{12}f(0) \\ f'(0) &= -B_{21}f(b) + B_{22}f(0) \end{aligned} \right\}. \end{aligned} \quad (7.10)$$

Note that $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^\Pi$. By Lemma 4.1(iv),

$$\Delta_{\tilde{A}'/\tilde{A}}^\Pi(z) = \frac{\det(B' - M(z))}{\det(B - M(z))}, \quad z \in \rho(\tilde{A}) \cap \rho(A_0). \quad (7.11)$$

If for some boundary triplet $\tilde{\Pi}$ the pair $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^{\tilde{\Pi}}$, then, by Proposition 4.5, there exist a constant $c \in \mathbb{C}$ such that

$$\Delta_{\tilde{A}'/\tilde{A}}^{\tilde{\Pi}}(z) = c \frac{\det(\tilde{B}' - M(z))}{\det(\tilde{B} - M(z))}, \quad z \in \rho(\tilde{A}) \cap \rho(A_0) \cap \rho(A'_0). \quad (7.12)$$

Slightly more complicated is the case when $\tilde{A}' = A_{B'}$ and $\tilde{A} = A_0$. From Corollary 4.4(i) we get the existence of a boundary triplet $\tilde{\Pi}$ such that $\{\tilde{A}', \tilde{A}\} \in \mathfrak{D}^{\tilde{\Pi}}$ which is regular. By Proposition 4.9(iii), there exist a constant $c \in \mathbb{C}$ and a real number $\mu \in \rho(B')$ such that

$$\Delta_{\tilde{A}'/\tilde{A}}^{\tilde{\Pi}}(z) = c \frac{\det(B' - M(z))}{\det(B' - \mu)}, \quad z \in \rho(\tilde{A}).$$

In particular, if $B' = 0$, then $\tilde{A}' = A_1$. Therefore chosen $\mu = 1$ we get

$$\Delta_{\tilde{A}'/\tilde{A}}^{\tilde{\Pi}}(z) = c \det(M(z)), \quad z \in \rho(\tilde{A}).$$

This yields

$$\Delta_{\tilde{A}'/\tilde{A}}^{\tilde{\Pi}}(z) = c \frac{\det C'(z, b)}{\det S(z, b)}, \quad z \in \rho(\tilde{A}),$$

which generalizes (7.4) to the case of a bounded interval.

Proposition 7.1. *Let A be the minimal Sturm-Liouville operator on $[0, b]$ defined by (7.6) and let $B \in \mathbb{C}^{2n \times 2n}$, $B_I := \text{Im}(B) \geq 0$, and $\ker B_I = \{0\}$. Let also $A_B = A^* \upharpoonright \text{dom}(A_B)$, $\text{dom}(A_B) := \ker(\Gamma_1 - B\Gamma_0)$. Then:*

- (i) A is simple.
- (ii) A_B is a m -dissipative and completely non-selfadjoint operator with discrete spectrum such that $\mathbb{R} \subseteq \rho(A_B)$. Additionally, A_B is complete.
- (iii) A_B belongs to the class C_0 . Moreover, the perturbation determinant $d(\cdot) = \Delta_{A_B/A_B^*}^\Pi(\cdot)$ is an annihilation function for A_B , that is, $d(A_B) = 0$.
- (iv) The annihilation function $d(\cdot)$ is minimal if and only if $\dim(\ker(B - M(z))) = 1$ for any $z \in \sigma(A_B) \cap \mathbb{C}_+ = \sigma_p(A_B)$.

Proof. (i) It follows immediately from the Cauchy uniqueness theorem.

(ii) The first claim follows from Proposition 5.22(i) and (ii). Further, it is well known that the resolvent of A_0 is of trace class. It follows from Proposition 2.6 that the resolvent of A_B is also of trace class. Since, in addition, A_B is m -dissipative, it follows from [32, Theorem V.6.1]) that A_B is complete.

(iii) and (iv) These statements follow from Proposition 5.22(iii) and (iv). \square

7.3 Second order elliptic operators in domains with compact boundary

7.3.1 Elliptic background

Here we present some known facts on second order elliptic operators, cf. [35] and [50], which we are need in the following. Consider the second-order formally symmetric elliptic operator with smooth real coefficients in a domain $\Omega \subset \mathbb{R}^n$ with smooth compact boundary,

$$\mathcal{A} := - \sum_{j,k=1}^n \frac{\partial}{\partial x_j} a_{jk}(x) \frac{\partial}{\partial x_j} + q(x), \quad a_{jk} = \bar{a}_{jk}, \quad q = \bar{q} \in C^\infty(\bar{\Omega}). \quad (7.13)$$

Recall that ellipticity of \mathcal{A} means that its (principle) symbol $a_0(x, \xi)$ satisfies

$$a_0(x, \xi) := \sum_{|\alpha|=|\beta|=1} a_{\alpha\beta}(x) \xi^{\alpha+\beta} \neq 0, \quad (x, \xi) \in \bar{\Omega} \times (\mathbb{R}^n \setminus \{0\}).$$

Let $A = A_{\min}$ be the minimal elliptic operator associated in $L^2(\Omega)$ with the expression (7.13). Due to Green's identity the minimal operator $A = A_{\min}$ is symmetric in $L^2(\Omega)$. Any proper extension $\tilde{A} \in \text{Ext}_A$ of A is called a realization of \mathcal{A} . Clearly, any realization \tilde{A} of \mathcal{A} is closable. We equip $\text{dom}(A_{\max})$ with the corresponding graph norm. It is known (cf. [6, 48]) for bounded domains that $\text{dom}(A_{\min}) = H_0^2(\Omega)$ where the graph norm of $\text{dom}(A_{\min})$ and the norm of the Sobolev space $H_0^2(\Omega)$ are equivalent. However, in contrast to the case of ordinary differential operators one has instead of $\text{dom}(A_{\max}) = H^2(\Omega)$ always

$$H^2(\Omega) \subset \text{dom}(A_{\max}) \subset H_{\text{loc}}^2(\Omega).$$

Since the symbol $a_0(x, \xi)$ is real the operator \mathcal{A} is properly elliptic (see [48]).

Hypothesis 7.2. *Let \mathcal{A} be a formally symmetric second order elliptic differential expression of the form (7.13) given on the bounded subset $\Omega \subseteq \mathbb{R}^n$ with smooth boundary $\partial\Omega$ which is uniformly elliptic. In addition, let $a_{\alpha\beta}(\cdot) \in C_b^2(\Omega)$ for $|\alpha| + |\beta| \leq 2$ and $a_{\alpha\beta}(\cdot) \in C_{ub}^\infty(\Omega)$, cf. notation at the end of the introduction, for $|\alpha| + |\beta| = 2$.*

In particular, assuming Hypothesis 7.2 we have $\text{dom}(A_{\min}) = H_0^2(\Omega)$. Notice that for bounded Ω any elliptic differential expression \mathcal{A} with $C(\overline{\Omega})$ -coefficients is automatically uniformly elliptic in $\overline{\Omega}$.

Denote by $\frac{\partial}{\partial\nu}$ the conormal derivative:

$$\frac{\partial}{\partial\nu} = \sum_{j,k=1}^n a_{jk}(x) \cos(n, x_j) \frac{\partial}{\partial x_k} \quad (7.14)$$

and set

$$G_0 u := \gamma_0 u := u|_{\partial\Omega}, \quad G_1 u := \gamma_0 \left(\frac{\partial u}{\partial\nu} \right) = \left(\frac{\partial u}{\partial\nu} \right) \Big|_{\partial\Omega}, \quad u \in \text{dom}(A_{\max}).$$

We define the Dirichlet and Neumann realizations \widehat{A}_{G_0} and \widehat{A}_{G_1} by setting

$$\begin{aligned} \widehat{A}_{G_j} &:= A_{\max} \upharpoonright \text{dom}(\widehat{A}_{G_j}), \\ \text{dom}(\widehat{A}_{G_j}) &:= \{u \in H^2(\Omega) \mid G_j u = 0\}, \quad j \in \{0, 1\}. \end{aligned} \quad (7.15)$$

It is well known that under Hypothesis 7.2 the realization \widehat{A}_{G_j} is selfadjoint in $H^0(\Omega) := L^2(\Omega)$, $\widehat{A}_{G_j} = \widehat{A}_{G_j}^*$, $j \in \{0, 1\}$.

To apply Proposition 4.9 and Corollary 4.10 we need a boundary triplet for A^* . Note, that the classical Green's formula reads now as follows

$$\begin{aligned} (\mathcal{A}u, v) - (u, \mathcal{A}v) &= \int_{\partial\Omega} \left(\frac{\partial u}{\partial\nu} \cdot \overline{v} - u \cdot \overline{\frac{\partial v}{\partial\nu}} \right) ds \\ &= \int_{\partial\Omega} \left(G_1 u \cdot \overline{G_0 v} - G_0 u \cdot \overline{G_1 v} \right) ds, \quad u, v \in H^2(\Omega). \end{aligned} \quad (7.16)$$

Proposition 7.3 ([35]). *Let the Hypothesis 7.2 be satisfied and let $0 \in \rho(\widehat{A}_{G_0})$. Then for any $s \in \mathbb{R}$ the operator G_0 isomorphically maps the set*

$$Z_{\mathcal{A}}^s(\Omega) := \{u \in H^s(\Omega) : A_{\max} u = 0\}$$

onto $H^{s-1/2}(\partial\Omega)$.

Definition 7.4 ([35, 63]). Assume Hypothesis 7.2.

(i) Let $z \in \rho(\widehat{A}_{G_0})$ and $\varphi \in H^{s-1/2}(\partial\Omega)$, $s \in \mathbb{R}$. Then one defines $P(z)\varphi$ to be the unique $u \in Z_{\mathcal{A}-zI_{L^2(\Omega)}}^s(\Omega)$ satisfying $G_0 u = \varphi$.

(ii) The Poincare-Steklov operator $\Lambda(z)$ is defined by

$$\Lambda(z) : H^{s-1/2}(\partial\Omega) \rightarrow H^{s-3/2}(\partial\Omega), \quad \Lambda(z)\varphi = G_1 P(z)\varphi. \quad (7.17)$$

To be precise we denote the operator $\Lambda(\cdot) : H^s(\partial\Omega) \rightarrow H^{s-1}(\partial\Omega)$ by $\Lambda_s(\cdot)$. Notice that $\Lambda_s(z) \in [H^s(\partial\Omega), H^{s-1}(\partial\Omega)]$.

Further, let $\Delta_{\partial\Omega}$ be the Laplace-Beltrami operator in $L^2(\partial\Omega)$. Since $\Delta_{\partial\Omega} \geq 0$, the operator $(-\Delta_{\partial\Omega} + I)^{-s/2}$ isomorphically maps $L^2(\partial\Omega)$ onto $H^s(\partial\Omega)$, $s \in \mathbb{R}$.

Notice that the classical Green formula (7.16) cannot be extended to $u, v \in \text{dom}(A^*)$ since the traces $G_0 u$ and $G_1 u$ belong to the spaces $H^{-1/2}(\partial\Omega)$ and $H^{-3/2}(\partial\Omega)$, respectively (see [48, 36]). A construction of a boundary triplet for A^* as well as the respective regularization of the Green formula (7.16) goes back to the classical papers by Vishik [63] and Grubb [35]. An adaptation of this construction to the case of boundary triplets in the sense of Definition 2.2 was done in [50]. First we recall a result from [50] that modifies and completes [35, Theorem 3.1.2]

Proposition 7.5 ([50, Proposition 5.1]). *Let the Hypothesis 7.2 be satisfied and let $0 \in \rho(\widehat{A}_{G_0})$. Then the following statements are valid:*

(i) *The totality $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\mathcal{H} := L^2(\partial\Omega)$ and*

$$\begin{aligned} \Gamma_0 u &:= (-\Delta_{\partial\Omega} + I)^{-1/4} G_0 u, \\ \Gamma_1 u &:= (-\Delta_{\partial\Omega} + I)^{1/4} (G_1 - \Lambda(0)G_0)u, \end{aligned} \quad u \in \text{dom}(A_{\max}), \quad (7.18)$$

forms a boundary triplet for A^ . In particular, the Green formula*

$$(A^* u, v)_{L^2(\Omega)} - (u, A^* v)_{L^2(\Omega)} = (\Gamma_1 u, \Gamma_0 v)_{L^2(\partial\Omega)} - (\Gamma_0 u, \Gamma_1 v)_{L^2(\partial\Omega)}, \quad (7.19)$$

$u, v \in \text{dom}(A^)$, holds and $A_0 := A^* \upharpoonright \ker(\Gamma_0) = \widehat{A}_{G_0}$.*

(ii) *The operator valued function $\Lambda_{-\frac{1}{2}}(z) - \Lambda_{-\frac{1}{2}}(0)$ has the regularity property*

$$\Lambda_{-\frac{1}{2}}(z) - \Lambda_{-\frac{1}{2}}(0) : H^{-1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega), \quad z \in \rho(\widehat{A}_{G_0}). \quad (7.20)$$

Moreover, $\Lambda_{-\frac{1}{2}}(z) - \Lambda_{-\frac{1}{2}}(0) \in [H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)]$ for any $z \in \rho(\widehat{A}_{G_0})$.

(iii) *The corresponding Weyl function is given by*

$$M(z) = (-\Delta_{\partial\Omega} + I)^{1/4} (\Lambda_{-\frac{1}{2}}(z) - \Lambda_{-\frac{1}{2}}(0)) (-\Delta_{\partial\Omega} + I)^{1/4}, \quad z \in \mathbb{C}_{\pm}. \quad (7.21)$$

In contrast to the mapping $\Gamma = \{\Gamma_0, \Gamma_1\} : \text{dom}(A_{\max}) \rightarrow L^2(\partial\Omega) \times L^2(\partial\Omega)$, the mapping

$$G = \{G_0, G_1\} : \text{dom}(A_{\max}) \rightarrow H^{-1/2}(\partial\Omega) \times H^{-3/2}(\partial\Omega)$$

is not surjective. The following statement describes the range $\text{ran}(G)$.

Corollary 7.6. *Let the assumptions of Proposition (7.5) be satisfied. Then for any pair $\{h_0, h_1\} \in H^{-1/2}(\partial\Omega) \times H^{-3/2}(\partial\Omega)$ the system $G_j f = h_j, j \in \{0, 1\}$, has a solution $f \in \text{dom}(A_{\max})$ if and only if*

$$h_1 - \Lambda_{-\frac{1}{2}}(0)h_0 \in H^{1/2}(\partial\Omega). \quad (7.22)$$

Proof. If condition (7.22) is satisfied, then it follows from Proposition (7.5)(i) and the surjectivity of $\Gamma = \{\Gamma_0, \Gamma_1\} : \text{dom}(A_{\max}) \rightarrow L^2(\partial\Omega) \times L^2(\partial\Omega)$ that the system

$$\begin{cases} \Gamma_0 f = (-\Delta_{\partial\Omega} + I)^{-1/4} h_0 \\ \Gamma_1 f = (-\Delta_{\partial\Omega} + I)^{1/4} (h_1 - \Lambda_{-\frac{1}{2}}(0)h_0) \end{cases}$$

has a (non-unique) solution $f \in \text{dom}(A_{\max})$. According to definition (7.18), f also satisfies the system $G_j f = h_j, j \in \{0, 1\}$.

Conversely, assume that there is a vector $f \in \text{dom}(A_{\max})$ satisfying $G_j f = h_j, j \in \{0, 1\}$. Hence,

$$h_1 - \Lambda_{-\frac{1}{2}}(0)h_0 = G_1 f - \Lambda_{-\frac{1}{2}}(0)G_0 f = (-\Delta_{\partial\Omega} + I)^{-1/4} \Gamma_1 f \in H^{1/2}(\partial\Omega)$$

which verifies (7.22). \square

Let \tilde{A} be any proper extension of $A := A_{\min}$. We set

$$\Xi_{\tilde{A}} := G\text{dom}(\tilde{A}) \subseteq H^{-1/2}(\partial\Omega) \times H^{-3/2}(\partial\Omega).$$

Clearly, $\Xi_{\tilde{A}}$ is not necessarily the graph of an operator.

Lemma 7.7. *Let the Hypothesis 7.2 be satisfied.*

(i) *Let \tilde{A} be a proper extension of $A := A_{\min}$. There exists an operator $K : H^{-1/2}(\partial\Omega) \rightarrow H^{-3/2}(\partial\Omega)$ such that $\Xi_{\tilde{A}} = \text{gr}(K)$ if and only if \tilde{A} and \hat{A}_{G_0} are disjoint.*

(ii) *Let \tilde{A}_1 and \tilde{A}_2 be proper extensions of $A := A_{\min}$ which are disjoint from \hat{A}_{G_0} . Let $\Xi_j := G\text{dom}(\tilde{A}_j) = \text{gr}(K_j), j = 1, 2$. If $K_1 = K_2$, then $\tilde{A}_1 = \tilde{A}_2$.*

Proof. (i) Assume that the extensions \tilde{A} and \hat{A}_{G_0} are disjoint. Let $\{h_0, h_1\} \in \Xi_{\tilde{A}} = G\text{dom}(\tilde{A})$, $h_0 \in H^{-1/2}(\partial\Omega)$, $h_1 \in H^{-3/2}(\partial\Omega)$. If $h_0 = 0$, then $0 = G_0 f$, $f \in \text{dom}(\tilde{A}) \subseteq \text{dom}(A_{\max})$. Hence $f \in \text{dom}(\hat{A}_{G_0})$ which yields $f \in \text{dom}(\tilde{A}) \cap \text{dom}(\hat{A}_{G_0})$. Since $\text{dom}(\tilde{A})$ and $\text{dom}(\hat{A}_{G_0})$ are disjoint we find $f \in \text{dom}(A)$. Hence $\Gamma_1 f = 0$ which implies $h_1 = 0$. Therefore, $\Xi_{\tilde{A}}$ is the graph of a linear operator $K : H^{-1/2}(\partial\Omega) \rightarrow H^{-3/2}(\partial\Omega)$.

Conversely, if \tilde{A} and \hat{A}_{G_0} are not disjoint, then there is a non-trivial $f \in \text{dom}(\tilde{A}) \cap \text{dom}(\hat{A}_{G_0})$ such that $f \notin \text{dom}(A_{\min})$. Notice that $h_0 = G_0 f = 0$. Since $\Xi_{\tilde{A}}$ is the graph of an operator we get $0 = h_1 = G_1 f$. Hence $f \in \text{dom}(A_{G_0}) \cap \text{dom}(A_{G_1})$ which yields $f \in \text{dom}(A_{\min})$.

(ii) Let $\{h_0, h_1\} \in \Xi_1 = \Xi_2$. Then there are elements $f_1 \in \text{dom}(\tilde{A}_1)$ and $f_2 \in \text{dom}(\tilde{A}_2)$ such that $h_0 = G_0 f_1 = G_0 f_2$ and $h_1 = G_1 f_1 = G_1 f_2$. Setting

$f := f_1 - f_2 \in \text{dom}(A_{\max})$ we find $G_0 f = 0$ and $G_1 f = 0$. Hence $f \in \text{dom}(A_{G_0}) \cap \text{dom}(A_{G_1}) = \text{dom}(A_{\min})$. Hence $f_1 = f_2 + f$ where $f_2 \in \text{dom}(\tilde{A}_2)$ and $f \in \text{dom}(A_{\min})$ which yields $f_1 \in \text{dom}(\tilde{A}_2)$. Similarly we prove $f_2 \in \text{dom}(\tilde{A}_2)$. Therefore $\text{dom}(\tilde{A}_2) = \text{dom}(\tilde{A}_1)$ which yields $\tilde{A}_1 = \tilde{A}_2$. \square

If \tilde{A} is a proper extension of A_{\min} which is disjoint from \hat{A}_{G_0} , then there is an operator $K : H^{-1/2}(\partial\Omega) \rightarrow H^{-3/2}(\partial\Omega)$ such that $G\text{dom}(\tilde{A}) = \text{gr}(K)$. The converse is in general not true. That means, not for every operator $K : H^{-1/2}(\partial\Omega) \rightarrow H^{-3/2}(\partial\Omega)$ there exists a proper extension \tilde{A} of A_{\min} such that $\text{gr}(K) = G\text{dom}(\tilde{A})$.

Lemma 7.8. *Let the Hypothesis 7.2 be satisfied and let $0 \in \rho(\hat{A}_{G_0})$. Further, let $K : H^{-1/2}(\partial\Omega) \rightarrow H^{-3/2}(\partial\Omega)$. There exists a unique proper extension \tilde{A} of A_{\min} such that \tilde{A} is disjoint from \hat{A}_{G_0} and $\text{gr}(K) = G\text{dom}(\tilde{A})$ if and only if the regularity condition $\text{ran}(K - \Lambda_{-\frac{1}{2}}(0)) \subseteq H^{1/2}(\partial\Omega)$ is satisfied.*

Proof. Let $\{h_0, h_1\} \in \text{gr}(K)$. Notice that $h_1 = Kh_0$. Using the assumption $\text{ran}(K - \Lambda_{-\frac{1}{2}}(0)) \subseteq H^{1/2}(\partial\Omega)$ we get $h_1 - \Lambda_{-\frac{1}{2}}(0)h_0 = Kh_0 - \Lambda_{-\frac{1}{2}}(0)h_0 \in H^{1/2}(\partial\Omega)$. By Corollary 7.6 there is an element $f \in \text{dom}(A_{\max})$ such that $h_j = G_j f$, $j = 0, 1$. We set

$$\text{dom}(\tilde{A}) := \{f \in \text{dom}(A_{\max}) : h_j = G_j f, j = 0, 1, \{h_0, h_1\} \in \text{gr}(K)\}.$$

Obviously, we have $\text{dom}(A_{\min}) \subseteq \text{dom}(\tilde{A}) \subseteq \text{dom}(A_{\max})$. Setting $\tilde{A} := A_{\max} \upharpoonright \text{dom}(\tilde{A})$ we define a proper extension of A_{\min} such that $\text{gr}(K) = G\text{dom}(\tilde{A})$. By Lemma 7.7(i) we get that \tilde{A} is disjoint from \hat{A}_{G_0} . The uniqueness follows from Lemma 7.7(i).

Conversely, if there is a proper extension \tilde{A} such that $G\text{dom}(\tilde{A}) = \text{gr}(K)$, then $h_j = G_j f$, $j = 0, 1$, $f \in \text{dom}(\tilde{A})$, $\{h_0, h_1\} \in \text{gr}(K)$. Applying Corollary 7.6 we get $h_1 - \Lambda_{-\frac{1}{2}}(0)h_0 \in H^{1/2}(\partial\Omega)$. Hence $Kh_0 - \Lambda_{-\frac{1}{2}}(0)h_0 \in H^{1/2}(\partial\Omega)$ for any $h_0 \in \text{dom}(K)$. Therefore $\text{ran}(K - \Lambda_{-\frac{1}{2}}(0)) \subseteq H^{1/2}(\partial\Omega)$. \square

Let $K : H^{-1/2}(\partial\Omega) \rightarrow H^{-3/2}(\partial\Omega)$. We set

$$\begin{aligned} \hat{A}_K &:= A_{\max} \upharpoonright \text{dom}(\hat{A}_K), \\ \text{dom}(\hat{A}_K) &:= \{f \in \text{dom}(A_{\max}) : G_1 f = KG_0 f\}. \end{aligned} \tag{7.23}$$

Obviously, \hat{A}_K is a proper extension of A_{\min} . Let \tilde{A} be a proper extension disjoint from \hat{A}_{G_0} and let K the operator defined by Lemma 7.7. Then a straightforward computation shows that $\tilde{A} = \hat{A}_K$. In general, for any operator $K : H^{-1/2}(\partial\Omega) \rightarrow H^{-3/2}(\partial\Omega)$ one has only $G\text{dom}(\hat{A}_K) \subseteq \text{gr}(K)$.

Corollary 7.9. *Let the assumptions of Lemma 7.8 be satisfied and let $K : H^{-1/2}(\partial\Omega) \rightarrow H^{-3/2}(\partial\Omega)$. The condition $\text{gr}(K) = G\text{dom}(\hat{A}_K)$ is satisfied if and only if $\text{ran}(K - \Lambda_{-\frac{1}{2}}(0)) \subseteq H^{1/2}(\partial\Omega)$.*

Proof. Let $\{h_0, h_1\} \in \text{gr}(K)$, then by the assumption $\text{ran}(K - \Lambda_{-\frac{1}{2}}(0))$ we get $h_1 - \Lambda_{-\frac{1}{2}}(0)h_0 \in H^{1/2}(\partial\Omega)$. By Corollary 7.6 there is a $f \in \text{dom}(A_{\max})$ such that $h_1 = G_1f$ and $h_0 = G_0f$. Since $G_1f = h_1 = Kh_0 = KG_0f$ we get $f \in \text{dom}(A_{\max})$. Hence $f \in \text{dom}(\hat{A}_K)$ and $\text{gr}(K) = G\text{dom}(\hat{A}_K)$.

Conversely, if $\text{gr}(K) = G\text{dom}(\hat{A}_K)$, then for any $\{h_0, h_1\} \in \text{gr}(K)$ there is $f \in \text{dom}(\hat{A}_K)$ such that $h_0 = G_0f$ and $h_1 = G_1f$. By Corollary 7.6 we get $h_1 - \Lambda_{-\frac{1}{2}}(0)h_0 = (K - \Lambda_{-\frac{1}{2}}(0))h_0 \in H^{1/2}(\partial\Omega)$ which yields $\text{ran}(K - \Lambda_{-\frac{1}{2}}(0)) \subseteq H^{1/2}(\partial\Omega)$. \square

Corollary 7.9 suggests besides the operator $K : H^{-1/2}(\partial\Omega) \longrightarrow H^{-3/2}(\partial\Omega)$ to consider its restriction $K' : H^{-1/2}(\partial\Omega) \longrightarrow H^{-3/2}(\partial\Omega)$ defined by

$$\begin{aligned} K' &:= K \upharpoonright \text{dom}(K'), \\ \text{dom}(K') &:= \{h \in \text{dom}(K) : Kh - \Lambda_{-\frac{1}{2}}(0)h \in H^{1/2}(\partial\Omega)\} \subseteq \text{dom}(K). \end{aligned} \quad (7.24)$$

Clearly, $\text{gr}(K') = G\text{dom}(\hat{A}_K)$, i.e. $\hat{A}_K = \hat{A}_{K'}$. For instance, if $\mathbb{O} : H^{-1/2}(\partial\Omega) \longrightarrow H^{-3/2}(\partial\Omega)$ is the zero operator, then obviously, $\hat{A}_{\mathbb{O}} = \hat{A}_{G_1}$. However, $\mathbb{O}' := \mathbb{O} \upharpoonright \text{dom}(\mathbb{O}')$,

$$\text{dom}(\mathbb{O}') := \{f \in H^{-1/2}(\partial\Omega) : -\Lambda_{-\frac{1}{2}}(0)f \in H^{1/2}(\partial\Omega)\} = H^{3/2}(\partial\Omega). \quad (7.25)$$

Hence $\hat{A}_{G_1} = \hat{A}_{\mathbb{O}'}$ and $\text{dom}(\hat{A}_{G_1}) = \{f \in H^2(\Omega) : G_1f = 0\}$.

Obviously, the proper extension \hat{A}_K admits a description with respect to the boundary triplet $\Pi = \{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ given by Proposition 7.5

Proposition 7.10 ([50, Proposition 3.8]). *Assume the conditions of Proposition 7.5. Let $K : H^{-1/2}(\partial\Omega) \longrightarrow H^{-3/2}(\partial\Omega)$ and let $\Pi = \{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ be the boundary triplet for A^* given by (7.18). Then the following holds:*

(i) $\hat{A}_K = A_{B_K}$, where $A_{B_K} := A^* \upharpoonright \ker(\Gamma_1 - B_K\Gamma_1)$ and

$$B_K := (-\Delta_{\partial\Omega} + I)^{1/4}(K' - \Lambda_{-\frac{1}{2}}(0))(-\Delta_{\partial\Omega} + I)^{1/4} : L^2(\partial\Omega) \longrightarrow L^2(\partial\Omega). \quad (7.26)$$

(ii) *The operator \hat{A}_K is closed and disjoint from \hat{A}_{G_0} if and only if the operator $K' - \Lambda_{-\frac{1}{2}}(0) : H^{-1/2}(\partial\Omega) \longrightarrow H^{1/2}(\partial\Omega)$ is closed.*

(iii) *Let \hat{A}_K be not closed (but necessary closable). Its closure is disjoint from \hat{A}_{G_0} if and only if the operator $K' - \Lambda_{-\frac{1}{2}}(0)$ is closable.*

(iv) *If $z \in \rho(\hat{A}_{G_0})$, then $z \in \rho(\hat{A}_K)$ if and only if the operator $K' - \Lambda_{-\frac{1}{2}}(z)$ maps $\text{dom}(K') \subset H^{-1/2}(\partial\Omega)$ onto $H^{1/2}(\partial\Omega)$ and its kernel is trivial.*

Proof. (i) From $\hat{A}_K = \hat{A}_{K'}$ and (7.24) we find that

$$G_1f - \Lambda_{-\frac{1}{2}}(0)G_0f = (K' - \Lambda_{-\frac{1}{2}}(0))G_0f, \quad f \in \text{dom}(\hat{A}_K),$$

which yields

$$\begin{aligned}\Gamma_1 f &= (-\Delta_{\partial\Omega} + I)^{1/2}(K' - \Lambda_{-\frac{1}{2}}(0))(-\Delta_{\partial\Omega} + I)^{1/2}(-\Delta_{\partial\Omega} + I)^{-1/2}G_0 f \\ &= (-\Delta_{\partial\Omega} + I)^{1/2}(K' - \Lambda_{-\frac{1}{2}}(0))(-\Delta_{\partial\Omega} + I)^{1/2}\Gamma_0 f, \quad f \in \text{dom}(\widehat{A}_K).\end{aligned}$$

Hence, if $f \in \text{dom}(\widehat{A}_K)$, then $f \in \ker(\Gamma_1 - B_K\Gamma_0)$. Therefore $\widehat{A}_K \subseteq A_{B_K}$.

Conversely, if $f \in \ker(\Gamma_1 - B_K\Gamma_0)$, then

$$(-\Delta_{\partial\Omega} + I)^{1/2}G_1 f = (-\Delta_{\partial\Omega} + I)^{1/2}(K' - \Lambda_{-\frac{1}{2}}(0))G_0 f, \quad f \in \text{dom}(A_{B_K}),$$

which implies $G_1 f = (K' - \Lambda_{-\frac{1}{2}}(0))G_0 f$, $f \in \text{dom}(B_K)$. Hence $\text{dom}(A_{B_K}) \subseteq \text{dom}(\widehat{A}_K)$. Consequently, $\text{dom}(\widehat{A}_K) = \text{dom}(A_{B_K})$ or $\widehat{A}_K = A_{B_K}$.

(ii) \widehat{A}_K is closed and disjoint from \widehat{A}_{G_0} if and only if B_K is closed. However, B_K is closed if and only if $K' - \Lambda_{-\frac{1}{2}}(0) : H^{-1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega)$ is closed.

(iii) The closure of \widehat{A}_K is disjoint from \widehat{A}_K if and only if B_K is closable. However, B_K is closable if and only if $K' - \Lambda_{-\frac{1}{2}}(0)$ is closable.

(iv) By Proposition 2.5 we get that $z \in \rho(\widehat{A}_K)$ if and only if $z \in \rho(B_K - M(z))$ where $M(z)$ is the Weyl function given by (7.21). Obviously, we have

$$B_K - M(z) = (-\Delta_{\partial\Omega} + I)^{1/4}(K' - \Lambda_{-\frac{1}{4}}(z))(-\Delta_{\partial\Omega} + I)^{1/4}. \quad (7.27)$$

However, the operator $B_K - M(z)$ is invertible if and only if the operator $K' - \Lambda_{-\frac{1}{2}}(z) : H^{-1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega)$ is invertible. \square

7.3.2 Perturbation determinants

To state the next result we recall the following definition.

Definition 7.11. Let $\mathcal{S}_p(\mathfrak{H}) = \{T \in \mathfrak{S}_\infty(\mathfrak{H}) : s_j(T) = O(j^{-1/p}), \text{ as } j \rightarrow \infty\}$, $p > 0$, where $s_j(T)$, $j \in \mathbb{N}$, denote the singular values of T (i.e., the eigenvalues of $(T^*T)^{1/2}$ decreasingly ordered counting multiplicity).

It is known that $\mathcal{S}_p(\mathfrak{H})$ is a two-sided (non-closed) ideal in $[\mathfrak{H}]$. Clearly, $\mathcal{S}_{p_1} \subset \mathcal{S}_{p_2}$ if $p_1 > p_2$. An important property of the classes $\mathcal{S}_p(\mathfrak{H})$ needed in the sequel is

$$\mathcal{S}_{p_1} \cdot \mathcal{S}_{p_2} \subset \mathcal{S}_p, \quad \text{where } p^{-1} = p_1^{-1} + p_2^{-1}. \quad (7.28)$$

Theorem 7.12 ([50, Theorem 4.13]). Assume the Hypothesis 7.2. Let $A_0 := \widehat{A}_{G_0}$ and $0 \in \rho(\widehat{A}_{G_0})$. Further, let $K : H^{-1/2}(\partial\Omega) \rightarrow H^{-3/2}(\partial\Omega)$ be an operator satisfying $\text{dom}(K) \subseteq L^2(\partial\Omega)$ and $\text{ran}(K) \subseteq L^2(\partial\Omega)$. Then

$$(\widehat{A}_K - z)^{-1} - (A_0 - z)^{-1} \in \mathcal{S}_{\frac{2n-2}{3}}(L^2(\Omega)), \quad z \in \rho(\widehat{A}_K) \cap \rho(A_0). \quad (7.29)$$

For $n = 2$ the resolvent difference in (7.29) is the trace class operator.

Lemma 7.13. *Let \mathfrak{X} and \mathfrak{Y} be Banach spaces and let $X : \mathfrak{X} \longrightarrow \mathfrak{Y}$ be a closed operator which is boundedly invertible. Further, let \mathfrak{X}_0 be another Banach space such that \mathfrak{X}_0 is a dense subset of \mathfrak{X} and the embedding $J : \mathfrak{X}_0 \longrightarrow \mathfrak{X}$ is continuous. If $\text{dom}(X) \subseteq J\mathfrak{X}_0$, then the operator $X_0 := XJ : \mathfrak{X}_0 \longrightarrow \mathfrak{Y}$, $\text{dom}(X_0) := \{f \in \mathfrak{X}_0 : Jf \in \text{dom}(X)\}$ is well defined, closed and boundedly invertible. In particular, $X^{-1} = JX_0^{-1}$.*

Proof. Let $\text{dom}(X_0) \ni f_n \longrightarrow f$ and $A_0 f_n \longrightarrow g \in \mathfrak{Y}$ as $n \rightarrow \infty$. Obviously we have $Jf_n \longrightarrow Jf$ and $XJf_n \longrightarrow g$ as $n \rightarrow \infty$. Since X is closed we get $Jf \in \text{dom}(X)$ and $XJf = g$. Hence $f \in \text{dom}(X_0)$ which shows that X_0 is closed. If $f \in \ker(X_0)$, then $XJf = 0$. hence $Jf = 0$ which yields $\ker(X_0) = \{0\}$. Finally, we have $\text{ran}(X) = \text{ran}(X_0) = \mathfrak{Y}$ which shows that X_0 is boundedly invertible. \square

In what follows we apply Lemma 7.13 with $\mathfrak{X} := H^{-1/2}(\partial\Omega)$, $\mathfrak{X}_0 := H^0(\partial\Omega)$, $\mathfrak{Y} := H^{1/2}(\partial\Omega)$ and $\mathfrak{Y}' := H^{-3/2}(\partial\Omega)$. Denote by J the embedding operator,

$$J : H^0(\partial\Omega) = L^2(\partial\Omega) \longrightarrow H^{-1/2}(\partial\Omega), \quad Jf = f. \quad (7.30)$$

Since $\text{dom}(K) \subseteq JL^2(\partial\Omega)$, we can set

$$K_0 := KJ : H^0(\partial\Omega) \longrightarrow H^{-3/2}(\partial\Omega), \\ \text{dom}(K_0) := \{f \in H^0(\partial\Omega) : Jf \in \text{dom}(K)\}.$$

Clearly, $\Lambda_0(z) = \Lambda_{-\frac{1}{2}}(z)J$, $\text{dom}(\Lambda_0(z)) := JH^0(\partial\Omega)$, and

$$K'_0 := K_0 \upharpoonright \text{dom}(K'_0), \\ \text{dom}(K'_0) := \{f \in \text{dom}(K_0) : (K_0 - \Lambda_0(0))f \in H^{1/2}(\partial\Omega)\}.$$

Clearly, $K'_0 = K'J : L^2(\partial\Omega) \longrightarrow H^{-3/2}(\partial\Omega)$.

Now we are in position to state the first main result of this section.

Proposition 7.14. *Let the assumptions of Theorem 7.12 be satisfied. Further, let $0 \in \rho(A_0) \cap \rho(\hat{A}_K)$. Then the following holds:*

(i) *For any $z \in \rho(\hat{A}_K) \cap \rho(A_0)$ the operator $K' - \Lambda_{-\frac{1}{2}}(z) : H^{-1/2}(\partial\Omega) \longrightarrow H^{1/2}(\partial\Omega)$ is boundedly invertible and*

$$(K' - \Lambda_{-\frac{1}{2}}(z))^{-1} \in \mathcal{S}_{\frac{2n-2}{3}}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)). \quad (7.31)$$

In particular, if $n = 2$ then

$$(K' - \Lambda_{-\frac{1}{2}}(z))^{-1} \in \mathfrak{S}_1(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)). \quad (7.32)$$

(ii) *Let $n = 2$. Then the boundary triplet $\tilde{\Pi} = \{\mathcal{H}, -\Gamma_1, \Gamma_0\}$, where $\mathcal{H} := L^2(\partial\Omega)$ and Γ_0, Γ_1 are given by (7.18), is regular for the pair $\{\hat{A}, A_0\}$, $\{\hat{A}_K, A_0\} \in \mathfrak{D}^{\tilde{\Pi}}$, and the perturbation determinant $\Delta_{\hat{A}_K/\hat{A}_0}^{\tilde{\Pi}}(\cdot)$ is*

$$\Delta_{\hat{A}_K/\hat{A}_0}^{\tilde{\Pi}}(z) = \det_{H^{-\frac{1}{2}}} \left(I - (K' - \Lambda_{-\frac{1}{2}}(0))^{-1} (\Lambda_{-\frac{1}{2}}(z) - \Lambda_{-\frac{1}{2}}(0)) \right) \\ = \det_{H^{\frac{1}{2}}} \left(I - (\Lambda_{-\frac{1}{2}}(z) - \Lambda_{-\frac{1}{2}}(0)) (K' - \Lambda_{-\frac{1}{2}}(0))^{-1} \right) \quad (7.33)$$

where $z \in \rho(\widehat{A}_K) \cap \rho(A_0)$.

(iii) Let $n = 2$. Then $(\Lambda_0(z) - \Lambda_0(0))(K'_0 - \Lambda_0(0))^{-1} \in \mathfrak{S}_1(H^{1/2}(\partial\Omega))$ and the perturbation determinant $\Delta_{\widehat{A}_K/A_0}^{\widetilde{\Pi}}(\cdot)$ admits the representation

$$\Delta_{\widehat{A}_K/A_0}^{\widetilde{\Pi}}(z) = \det_{H^{1/2}}(I - (\Lambda_0(z) - \Lambda_0(0))(K'_0 - \Lambda_0(0))^{-1}). \quad (7.34)$$

Proof. (i) By Proposition 2.5(i), $0 \in \rho(B_K - M(z))$ for any $z \in \rho(\widehat{A}_K) \cap \rho(A_0)$. Moreover, combining Proposition 2.6 with Theorem 7.12 we get $(B_K - M(z))^{-1} \in \mathcal{S}_{\frac{2n-2}{3}}(H^0(\partial\Omega))$. Combining this fact with (7.27) we obtain

$$\begin{aligned} (B_K - M(z))^{-1} \\ = (I - \Delta_{\partial\Omega})^{-1/4}(K' - \Lambda_{-\frac{1}{2}}(z))^{-1}(I - \Delta_{\partial\Omega})^{-1/4} \in \mathcal{S}_{\frac{2n-2}{3}}(H^0(\partial\Omega)). \end{aligned}$$

Since $(I - \Delta_{\partial\Omega})^{-1/4}$ isomorphically maps $H^s(\partial\Omega)$ onto $H^{s+1/2}(\partial\Omega)$ for $s \in \mathbb{R}$ we arrive at (7.31). Further, for $n = 2$ inclusion (7.31) implies

$$(K' - \Lambda_{-\frac{1}{2}}(z))^{-1} \in \mathcal{S}_{\frac{2}{3}}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)) \subset \mathfrak{S}_1(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)).$$

This proves the last statement.

(ii) By Theorem 7.12, $(\widehat{A}_K - z)^{-1} - (A_0 - z)^{-1} \in \mathfrak{S}_1(L^2(\Omega))$ since $n = 2$. Further, by Proposition 2.5(i), the condition $0 \in \rho(A_0) \cap \rho(\widehat{A}_K)$ is equivalent to $0 \in \rho(B_K - M(0)) = \rho(B_K)$. By Proposition 4.9(iii), a boundary triplet $\widetilde{\Pi} = \{\mathcal{H}, -\Gamma_1, \Gamma_0\}$ is regular for the pair $\{\widehat{A}_K, A_0\}$, hence $\{\widehat{A}_K, A_0\} \in \mathfrak{D}^{\widetilde{\Pi}}$, and the perturbation determinant $\Delta_{\widehat{A}_K/A_0}^{\widetilde{\Pi}}(\cdot)$ is given by (4.17) with $\mu = 0$ and constant $c = 1$ (see formula (4.18)). Inserting in this expression formulas (7.21) and (7.26) we arrive at the following formula for the determinant

$$\begin{aligned} \Delta_{\widehat{A}_K/A_0}^{\widetilde{\Pi}}(z) &= \det_{L^2(\partial\Omega)}(I - B_K^{-1}M(z)) = \\ &= \det_{H^0}\left(I - (I - \Delta_{\partial\Omega})^{-1/4}(K' - \Lambda_{-\frac{1}{2}}(0))^{-1}(\Lambda_{-\frac{1}{2}}(z) - \Lambda_{-\frac{1}{2}}(0))(I - \Delta_{\partial\Omega})^{1/4}\right) \end{aligned}$$

for $z \in \rho(\widehat{A}_K) \cap \rho(A_0)$. Further, according to Proposition 7.5(ii), $\Lambda_{-\frac{1}{2}}(z) - \Lambda_{-\frac{1}{2}}(0) \in [H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)]$. Combining this inclusion with (7.32) we get $T_2(z) \in \mathfrak{S}_1(H^0(\partial\Omega), H^{-1/2}(\partial\Omega))$ where

$$T_2(z) := (K' - \Lambda_{-\frac{1}{2}}(0))^{-1}(\Lambda_{-\frac{1}{2}}(z) - \Lambda_{-\frac{1}{2}}(0))(I - \Delta_{\partial\Omega})^{1/4}, \quad z \in \rho(\widehat{A}_K) \cap \rho(A_0).$$

Noting that $T_1 = (I - \Delta_{\partial\Omega})^{-1/4}$ isomorphically maps $H^{-1/2}(\partial\Omega)$ onto $H^0(\partial\Omega)$ we see that $T_2(z)T_1$ is well defined and $T_2(z)T_1 \in \mathfrak{S}_1(H^{-1/2}(\partial\Omega))$. Moreover, due to the inclusion (7.32), $T_1T_2(z) \in \mathfrak{S}_1(H^0(\partial\Omega))$. Taking both last inclusions into account and applying (A.1) we arrive at the equality

$$\Delta_{\widehat{A}_K/A_0}^{\widetilde{\Pi}}(z) = \det_{L^2(\partial\Omega)}(I - T_1T_2(z)) = c \det_{H^{-1/2}(\partial\Omega)}(I - T_2(z)T_1)$$

coinciding with the first identity in (7.33). The second identity in (7.33) is implied by combining the first one with the property (A.1). Note that the applicability of (A.1) is possible due to inclusion (7.32) and Proposition 7.5(ii).

(iii) By Lemma 7.13, the operator $(K'_0 - \Lambda_0(0))^{-1} : H^{1/2}(\partial\Omega) \longrightarrow L^2(\partial\Omega)$ is bounded and

$$(K' - \Lambda_{-\frac{1}{2}}(0))^{-1} = J(K'_0 - \Lambda_0(0))^{-1}. \quad z \in \rho(\widehat{A}_K) \cap \rho(A_0). \quad (7.35)$$

Combining this formula with (7.32) we get $J(K'_0 - \Lambda_0(0))^{-1} \in \mathfrak{S}_1(H^{1/2}(\partial\Omega))$. Therefore inserting (7.35) into the second formula in (7.33) we get

$$\Delta_{\widehat{A}_K/A_0}^{\widetilde{\Pi}}(z) = c \det_{H^{1/2}} \left(I - \left(\Lambda_{-\frac{1}{2}}(z) - \Lambda_{-\frac{1}{2}}(0) \right) J(K'_0 - \Lambda_0(0))^{-1} \right).$$

To arrive at (7.34) it remains to note that $\Lambda_{-\frac{1}{2}}(z)J = \Lambda_0(z)$. \square

Combining the chain rule 4.9 with Proposition 7.14 one arrives at the following statement.

Corollary 7.15. *Assume the Hypothesis 7.2. Let $A_0 := \widehat{A}_{G_0}$ and $0 \in \rho(\widehat{A}_{G_0})$. Further, let $K_j : H^{-1/2}(\partial\Omega) \longrightarrow H^{-3/2}(\partial\Omega)$ be an operator satisfying $\text{dom}(K_j) \subseteq L^2(\partial\Omega)$ and $\text{ran}(K_j) \subseteq L^2(\partial\Omega)$, and let $A_j := A_{K_j}$, $j \in \{1, 2\}$. Assume also that $0 \in \rho(A_0) \cap \rho(\widehat{A}_{K_j})$, $j \in \{1, 2\}$. Then the boundary triplet $\widetilde{\Pi} = \{\mathcal{H}, -\Gamma_1, \Gamma_0\}$ for A_{\max} with $\mathcal{H} := L^2(\partial\Omega)$ and Γ_0, Γ_1 given by (7.18), is regular for the family $\{\widehat{A}_{K_1}, \widehat{A}_{K_2}, A_0\}$, and the perturbation determinant is*

$$\Delta_{\widehat{A}_2/\widehat{A}_1}^{\Pi}(z) = \frac{\det_{H^{-\frac{1}{2}}} \left(I - (K'_2 - \Lambda_{-\frac{1}{2}}(0))^{-1} (\Lambda_{-\frac{1}{2}}(z) - \Lambda_{-\frac{1}{2}}(0)) \right)}{\det_{H^{-\frac{1}{2}}} \left(I - (K'_1 - \Lambda_{-\frac{1}{2}}(0))^{-1} (\Lambda_{-\frac{1}{2}}(z) - \Lambda_{-\frac{1}{2}}(0)) \right)}, \quad (7.36)$$

$$z \in \rho(\widehat{A}_{K_1}) \cap \rho(\widehat{A}_{K_2}) \cap \rho(A_0).$$

Our next goal is to show that under additional restrictions on K the perturbation determinant $\Delta_{\widehat{A}_K/A_0}^{\widetilde{\Pi}}(\cdot)$ can be computed in $L^2(\partial\Omega)$. To this end we introduce the operator-valued function $\Lambda_{0,0}(\cdot) : L^2(\partial\Omega) \longrightarrow L^2(\partial\Omega)$ by setting

$$\begin{aligned} \Lambda_{0,0}(z) &:= \Lambda_0(z) \upharpoonright \text{dom}(\Lambda_{0,0}(z)), \\ \text{dom}(\Lambda_{0,0}(z)) &:= \{f \in \text{dom}(\Lambda_0(z)) : \Lambda_0(z)f \in L^2(\partial\Omega)\} \end{aligned} \quad (7.37)$$

Lemma 7.16. *Let $0 \in \rho(\widehat{A}_{G_0})$. Then*

$$\text{dom}(\Lambda_{0,0}(z)) = H^1(\partial\Omega), \quad z \in \rho(\widehat{A}_{G_0}), \quad (7.38)$$

and, for any $z \in \rho(\widehat{A}_{G_0}) \cap \rho(\widehat{A}_{G_1})$ the operator $(\Lambda_{0,0}(z))^{-1}$ exists and satisfies $(\Lambda_{0,0}(z))^{-1} \in \mathcal{S}_1(H^0(\partial\Omega))$. Moreover, if $0 \in \rho(\widehat{A}_{G_0}) \cap \rho(\widehat{A}_{G_1})$, then the operator $\Lambda_{0,0}(0)$ is selfadjoint, has discrete spectrum, and $(\Lambda_{0,0}(0))^{-1} \in \mathcal{S}_1(H^0(\partial\Omega))$.

Proof. It follows from Definition 7.4 that $\text{dom}(\Lambda_{0,0}(\cdot)) \supseteq H^1(\partial\Omega)$. Let us prove the equality (7.38). Since both realizations \widehat{A}_{G_0} and \widehat{A}_{G_1} are self-adjoint, $\rho(\widehat{A}_{G_0}) \cap \rho(\widehat{A}_{G_1}) \supset \mathbb{C}_\pm$. Let $z \in \rho(\widehat{A}_{G_0}) \cap \rho(\widehat{A}_{G_1})$. Then $\text{dom}(\Lambda_1(z)) = H^1(\partial\Omega)$ and Λ_1 isomorphically maps $H^1(\partial\Omega)$ onto $H^0(\partial\Omega)$ (see [35, Theorem 5.2]). Since $\Lambda_{0,0}(z)h = \Lambda_1(z)h$ for $h \in H^1(\partial\Omega)$ we conclude that $\text{dom}(\Lambda_{0,0}(z)) = H^1(\partial\Omega)$ and $\text{ran}(\Lambda_{0,0}(z)) = H^0(\partial\Omega)$.

Next let $x_0 = \bar{x}_0 \in \rho(\widehat{A}_{G_0}) \setminus \rho(\widehat{A}_{G_1})$. We can assume without loss of generality that $x_0 = 0$. Otherwise we replace the expression \mathcal{A} by $\mathcal{A} - x_0 I$. Then, by Proposition 7.5, the difference $T(z) := \Lambda_{-\frac{1}{2}}(z) - \Lambda_{-\frac{1}{2}}(0) : H^{-1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega)$ is bounded. Hence the difference $\Lambda_{0,0}(z) - \Lambda_{0,0}(0)$ being a restriction of $T(z)$ is bounded in $H^0(\partial\Omega)$ and $\text{dom}(\Lambda_{0,0}(0)) = \text{dom}(\Lambda_{0,0}(z)) = H^1(\partial\Omega)$ for $z \in \rho(\widehat{A}_{G_0})$. Further, since $\text{ran}(\Lambda_{0,0}(z))^{-1} = H^1(\partial\Omega)$ for $z \in \rho(\widehat{A}_{G_0}) \cap \rho(\widehat{A}_{G_1})$, we have $(\Lambda_{0,0}(z))^{-1} \in \mathcal{S}_1(H^0(\partial\Omega))$.

Clearly, for any $x_0 = \bar{x}_0 \in \rho(\widehat{A}_{G_0})$ the operator $\Lambda_{0,0}(x_0)$ is symmetric. If, in addition, $x_0 \in \rho(\widehat{A}_{G_1})$, then the operator $\Lambda_{0,0}(x_0)$ is selfadjoint since $\text{ran}(\Lambda_{0,0}(x_0)) = H^0(\partial\Omega)$. If $0 \in \rho(\widehat{A}_{G_0}) \setminus \rho(\widehat{A}_{G_1})$, then the self-adjointness of $\Lambda_{0,0}(0)$ is implied by the self-adjointness of $\Lambda_{0,0}(x_0)$ with $x_0 = \bar{x}_0 \in \rho(\widehat{A}_{G_0}) \cap \rho(\widehat{A}_{G_1})$ and the boundedness of $\Lambda_{0,0}(x_0) - \Lambda_{0,0}(0)$ in $H^0(\partial\Omega)$.

Further, since the boundary $\partial\Omega$ is compact, the spectrum of $\Lambda_{0,0}(0)$ is discrete. Moreover, since by (7.38), $\text{ran}(\Lambda_{0,0}(z))^{-1} = H^1(\partial\Omega)$ for $\rho(\widehat{A}_{G_0}) \cap \rho(\widehat{A}_{G_1})$, we have $(\Lambda_{0,0}(z))^{-1} \in \mathcal{S}_1(H^0(\partial\Omega))$. \square

Proposition 7.17. *Assume the Hypothesis 7.2. Let $K : H^{-1/2}(\partial\Omega) \rightarrow H^{-3/2}(\partial\Omega)$ be an operator satisfying $\text{dom}(K) \subseteq L^2(\partial\Omega)$ and $\text{ran}(K) \subseteq L^2(\partial\Omega)$. Assume also that $0 \in \rho(\widehat{A}_{G_0}) \cap \rho(\widehat{A}_{G_1}) \cap \rho(\widehat{A}_K)$ and*

$$\widehat{K}_0 := KJ : L^2(\partial\Omega) \rightarrow L^2(\partial\Omega), \quad \text{dom}(K) = J\text{dom}(\widehat{K}_0), \quad (7.39)$$

where J is the embedding operator given by (7.30). If \widehat{K}_0 is relatively compact with respect to $\Lambda_{0,0}(0)$, then

$$(\Lambda_{0,0}(z) - \Lambda_{0,0}(0))(\widehat{K}_0 - \Lambda_{0,0}(0))^{-1} \in \mathcal{S}_{\frac{1}{2}}(H^0(\partial\Omega)) \subset \mathfrak{S}_1(H^0(\partial\Omega)), \quad (7.40)$$

$z \in \rho(\widehat{A}_K) \cap \rho(\widehat{A}_{G_0})$, and the perturbation determinant $\Delta_{\widehat{A}_K/A_0}^{\widehat{\Pi}}(\cdot)$ given by (7.33) admits the representation

$$\Delta_{\widehat{A}_K/A_0}^{\widehat{\Pi}}(z) = \det_{L^2} \left(I - (\Lambda_{0,0}(z) - \Lambda_{0,0}(0))(\widehat{K}_0 - \Lambda_{0,0}(0))^{-1} \right), \quad (7.41)$$

for $z \in \rho(\widehat{A}_K) \cap \rho(\widehat{A}_{G_0})$. In particular, representation (7.41) holds whenever \widehat{K}_0 is bounded, i.e. $\widehat{K}_0 \in [H^0(\partial\Omega)]$.

Proof. (i) Let us prove the inclusion (7.40). According to (7.20), $\Lambda_0(z) - \Lambda_0(0) : H^0(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega)$. Hence

$$\overline{\Lambda_0(z) - \Lambda_0(0)} \in \mathcal{S}_2(H^0(\partial\Omega)), \quad z \in \rho(A_0). \quad (7.42)$$

Further, by definition (7.30), J^* continuously embeds $H^{1/2}(\partial\Omega)$ into $L^2(\partial\Omega)$. Therefore

$$J^*(\Lambda_0(z) - \Lambda_0(0))h = (\Lambda_{0,0}(z) - \Lambda_{0,0}(0))h, \quad h \in H^1(\partial\Omega). \quad (7.43)$$

Combining relations (7.43) and (7.42), using $J^* \in S_2(H^{1/2}, H^0)$ and taking property (7.28) into account we obtain

$$\overline{\Lambda_{0,0}(z) - \Lambda_{0,0}(0)} \in \mathcal{S}_1(H^0(\partial\Omega)), \quad z \in \rho(A_0). \quad (7.44)$$

On the other hand, by Lemma 7.16 the operator $\Lambda_{0,0}(0)$ is selfadjoint, and due to the assumption $0 \in \rho(\widehat{A}_{G_1})$, $\Lambda_{0,0}(0)$ is invertible and $(\Lambda_{0,0}(0))^{-1} \in \mathcal{S}_1(H^0(\partial\Omega))$.

Further, by Proposition 7.10(iv) the inclusion $0 \in \rho(\widehat{A}_K)$ is equivalent to the inclusion $0 \in \rho(\widehat{K}_0 - \Lambda_{0,0}(0))$. Hence $0 \in \rho(I - \widehat{K}_0(\Lambda_{0,0}(0))^{-1})$. Thus, the inverse operator $(I - \widehat{K}_0(\Lambda_{0,0}(0))^{-1})^{-1} \in [H^0(\partial\Omega)]$ and

$$(\widehat{K}_0 - \Lambda_{0,0}(0))^{-1} = -(\Lambda_{0,0}(0))^{-1} \left(I - \widehat{K}_0(\Lambda_{0,0}(0))^{-1} \right)^{-1} \in \mathcal{S}_1(H^0(\partial\Omega)).$$

Using (7.44) and taking into account (7.28) we arrive at (7.40).

(ii) In this step we prove formula (7.41). Since $\text{ran}(K) \subseteq L^2(\partial\Omega)$, the operator $K_0 = KJ : H^0(\partial\Omega) \rightarrow H^{-3/2}(\partial\Omega)$ satisfies $\text{ran}(K_0) \subseteq L^2(\partial\Omega)$. However, we distinguish it from the operator \widehat{K}_0 defined by (7.39). Since $\text{dom}(\widehat{K}_0) = \text{dom}(K_0)$ and $\text{ran}(\widehat{K}_0) \subseteq L^2(\partial\Omega)$, one gets from definition (7.24) and (7.37)

$$\text{dom}(K'_0) := \{h \in \text{dom}(\widehat{K}_0) \cap H^1(\partial\Omega) : \widehat{K}_0 h - \Lambda_{0,0}(0)h \in H^{1/2}(\partial\Omega)\},$$

and in accordance with (7.30)

$$J^*(K'_0 - \Lambda_0(0))h = (\widehat{K}_0 - \Lambda_{0,0}(0))h, \quad h \in \text{dom}(K'_0). \quad (7.45)$$

Clearly, $J^*(K'_0 - \Lambda_0(0)) \subseteq \widehat{K}_0 - \Lambda_{0,0}(0)$ (in fact, the inclusion is always strict). Why? Since \widehat{K}_0 is relatively compact with respect to $\Lambda_{0,0}(0)$, $\text{dom}(\widehat{K}_0 - \Lambda_{0,0}(0)) = \text{dom}(\Lambda_{0,0}(0))$. Moreover, since $0 \in \rho(\widehat{A}_{K_0})$ and $\text{dom}(K') \subseteq H^0(\partial\Omega)$, Proposition 7.10(iv) yields $\text{ran}(K'_0 - \Lambda_0(0)) = \text{ran}(K' - \Lambda_{-\frac{1}{2}}(0)) = H^{1/2}(\partial\Omega)$. Hence the range $\text{ran}(\widehat{K}_0 - \Lambda_{0,0}(0))$ is dense in $H^0(\partial\Omega)$.

On the other hand, by Lemma 7.16, the operator $\Lambda_{0,0}(0)$ is selfadjoint and its spectrum is discrete. In particular, $\Lambda_{0,0}(0)$ is a Fredholm operator with zero index. In turn, since \widehat{K}_0 is $\Lambda_{0,0}(0)$ -compact, the operator $\widehat{K}_0 - \Lambda_{0,0}(0)$ is also Fredholm operator with zero index [37, Theorem 4.5.26] (in fact, it has discrete spectrum too). Therefore the range $\text{ran}(\widehat{K}_0 - \Lambda_{0,0}(0))$ is closed and being dense in $H^0(\partial\Omega)$, coincides with $H^0(\partial\Omega)$. Since $\text{ind}(\widehat{K}_0 - \Lambda_{0,0}(0)) = 0$, the latter is equivalent to $0 \in \rho(\widehat{K}_0 - \Lambda_{0,0}(0))$. From Lemma 7.13 we get the existence of $(K'_0 f - \Lambda_0(0))^{-1} : H^{1/2}(\partial\Omega) \rightarrow H^0(\partial\Omega)$. Combining this fact with (7.45) we find

$$(\widehat{K}_0 - \Lambda_{0,0}(0))^{-1} J^* = (K'_0 - \Lambda_0(0))^{-1} \quad (7.46)$$

Inserting (7.46) into (7.34) we obtain

$$\Delta_{\widehat{A}_K/A_0}^{\widetilde{\Pi}}(z) = c \det_{H^{1/2}} \left(I - (\Lambda_0(z) - \Lambda_0(0))(\widehat{K}_0 - \Lambda_{0,0}(0))^{-1} J^* \right).$$

Using (7.40) we get by the cyclicity property (see (A.1)) that

$$\Delta_{\widehat{A}_K/A_0}^{\widetilde{\Pi}}(z) = c \det_{H^0} \left(I - J^*(\Lambda_0(z) - \Lambda_0(0))(\widehat{K}_0 - \Lambda_{0,0}(0))^{-1} \right),$$

$z \in \rho(\widehat{A}_K) \cap \rho(A_0)$. Combining this identity with (7.43) we arrive at (7.41). \square

Corollary 7.18. *Assume the Hypothesis 7.2. Let $K_j : H^{-1/2}(\partial\Omega) \rightarrow H^{-3/2}(\partial\Omega)$ be an operator satisfying $\text{dom}(K_j) \subseteq L^2(\partial\Omega)$ and $\text{ran}(K_j) \subseteq L^2(\partial\Omega)$, $j \in \{1, 2\}$. Further, let $0 \in \rho(\widehat{A}_{G_0}) \cap \rho(\widehat{A}_{G_1}) \cap \rho(\widehat{A}_{K_j})$, and*

$$\widehat{K}_{j,0} := K_j J : L^2(\partial\Omega) \rightarrow L^2(\partial\Omega), \quad \text{dom}(K_j) = J \text{dom}(\widehat{K}_0), \quad j \in \{1, 2\}.$$

If the operator $\widehat{K}_{j,0}$ is relatively compact with respect to $\Lambda_{0,0}(0)$, then the perturbation determinant $\Delta_{\widehat{A}_2/\widehat{A}_1}^{\Pi}(\cdot)$ given by (7.36) admits the representation

$$\Delta_{\widehat{A}_{K_2}/\widehat{A}_{K_1}}^{\Pi}(z) = \frac{\det_{L^2} \left(I - (\Lambda_{0,0}(z) - \Lambda_{0,0}(0))(\widehat{K}_{2,0} - \Lambda_{0,0}(0))^{-1} \right)}{\det_{L^2} \left(I - (\Lambda_{0,0}(z) - \Lambda_{0,0}(0))(\widehat{K}_{1,0} - \Lambda_{0,0}(0))^{-1} \right)},$$

for $z \in \rho(\widehat{A}_{K_2}) \cap \rho(\widehat{A}_{K_1}) \cap \rho(\widehat{A}_{G_0})$.

Proof. The proof is immediate by combining Corollary 7.15 with Proposition 7.17. \square

Consider Robin-type realizations

$$\begin{aligned} \widehat{A}_\sigma &:= A_{\max} \upharpoonright \text{dom}(\widehat{A}_\sigma), \\ \text{dom}(\widehat{A}_\sigma) &:= \{f \in H^2(\Omega) : G_1 f = \sigma G_0 f\}. \end{aligned}$$

It follows from the classical a priori estimate (see [4, Theorem 15.2]) that the realization \widehat{A}_σ is closed whenever $\sigma \in C^2(\partial\Omega)$. Moreover, in this case $\rho(\widehat{A}_\sigma) \neq \emptyset$ and \widehat{A}_σ is selfadjoint whenever σ is real.

Corollary 7.19. *Assume the conditions of Proposition 7.17. Let $\sigma \in C^2(\partial\Omega)$ and let $\widehat{\sigma}$ denote the multiplication operator induced by σ in $L^2(\partial\Omega)$. If $0 \in \rho(\widehat{A}_\sigma) \cap \rho(\widehat{A}_{G_0}) \cap \rho(\widehat{A}_{G_1})$, then the boundary triplet $\widetilde{\Pi} = \{\mathcal{H}, -\Gamma_1, \Gamma_0\}$ given in Proposition 7.14, is regular for the pair $\{\widehat{A}_\sigma, \widehat{A}_{G_0}\}$, $\{\widehat{A}_\sigma, \widehat{A}_{G_0}\} \in \mathfrak{D}^{\widetilde{\Pi}}$, and the corresponding perturbation determinant $\Delta_{\widehat{A}_\sigma/A_0}^{\widetilde{\Pi}}(\cdot)$ is*

$$\Delta_{\widehat{A}_\sigma/A_0}^{\widetilde{\Pi}}(z) = \det_{L^2(\partial\Omega)} \left(I - (\Lambda_{0,0}(z) - \Lambda_{0,0}(0))(\widehat{\sigma} - \Lambda_{0,0}(0))^{-1} \right),$$

$z \in \rho(\widehat{A}_\sigma) \cap \rho(\widehat{A}_{G_0})$.

Proof. Setting $K = \hat{\sigma}$ and noting that $\sigma \in C^2(\Omega)$ we easily get from (1.11)

$$\text{dom}(K - \Lambda_{-1/2}(0)) = \text{dom}(K) = \text{dom}(\hat{\sigma}) \subset \text{ran}(G_0) = H^{3/2}(\partial\Omega).$$

Since $\Lambda_{3/2}(0)$ is a restriction of $\Lambda_{-1/2}(0)$ one gets from (7.17) that $\text{ran}(\Lambda_{-1/2}(0) \upharpoonright H^{3/2}(\partial\Omega)) \subset H^{1/2}(\partial\Omega)$. Further, the assumption $\sigma \in C^2(\Omega)$ yields $\text{ran}(K \upharpoonright H^{3/2}(\partial\Omega)) \subset H^{3/2}(\partial\Omega)$. Combining these inclusions we arrive at the regularity property $\text{ran}(K - \Lambda_{-1/2}(0)) \subset H^{1/2}(\partial\Omega)$ (see (7.22)).

Hence $K' = K$ (see definition (7.24)) and $\text{dom}(\hat{A}_K) = \text{dom}(\hat{A}_\sigma)$. Moreover, since $\text{dom}(K), \text{ran}(K) \subset H^{3/2}(\partial\Omega)$, then according to (7.39), $\hat{K}_0 = \hat{\sigma}$. Finally, since $\hat{K}_0 = \hat{\sigma} \in [H^0(\partial\Omega)]$, one completes the proof by applying Proposition 7.17. \square

Appendix

A Infinite determinants

Let us briefly recall the definition of determinants and their basic properties following [32].

Definition A.1. Let T be a trace class operator, i.e. $T \in \mathfrak{S}_1(\mathcal{H})$, and let $\{\lambda_j(T)\}_{j=1}^\infty$ be its eigenvalues counted with respect to their algebraic multiplicity. The determinant $\det(I + T)$ is defined by $\det(I + T) := \prod_{j=1}^\infty (1 + \lambda_j(T))$.

The perturbation determinant has the following interesting properties.

Proposition A.2 ([32, Section 4.1]). *Let $T_1 \in [\mathcal{H}_1, \mathcal{H}_2]$ and $T_2 \in [\mathcal{H}_2, \mathcal{H}_1]$.*

(i) *If $T_1 T_2 \in \mathfrak{S}_1(\mathcal{H}_2)$ and $T_2 T_1 \in \mathfrak{S}_1(\mathcal{H}_1)$, then*

$$\det_{\mathcal{H}_2}(I + T_1 T_2) = \det_{\mathcal{H}_1}(I + T_2 T_1). \quad (\text{A.1})$$

(ii) *If $\mathcal{H} := \mathcal{H}_1 = \mathcal{H}_2$ and $T_1, T_2 \in \mathfrak{S}_1(\mathcal{H})$, then*

$$\det[(I + T_1)(I + T_2)] = \det(I + T_1) \cdot \det(I + T_2). \quad (\text{A.2})$$

(iii) *If $T \in \mathfrak{S}_1(\mathcal{H})$, then*

$$\det(I + T^*) = \overline{\det(I + T)}.$$

For technical reasons we need a slightly improved version of the property (A.1).

Lemma A.3. *Let $T = T^* \geq 0$ such that $0 \in \rho(T)$. Further, let C be linear operator such that $\text{dom}(C) \supseteq \text{ran}(T)$. If $\overline{T^{-1}C} \in \mathfrak{S}_1(\mathfrak{H})$ and $CT^{-1} \in \mathfrak{S}_1(\mathfrak{H})$, then*

$$\det(I + \overline{T^{-1}C}) = \det(I + CT^{-1}). \quad (\text{A.3})$$

Proof. Let $K = K^*$ be a bounded operator such that $\|K\| < \inf \sigma(T)$. In this case we have $T + K \geq 0$ and $0 \in \rho(T + K)$. Moreover, we find $\overline{(T + K)^{-1}C} \in \mathfrak{S}_1(\mathfrak{H})$ and $C(T + K)^{-1} \in \mathfrak{S}_1(\mathfrak{H})$. Both relations immediately follow from the identities

$$(T + K)^{-1} = (I - (T + K)^{-1}K)T^{-1} = T^{-1}(I - K(T + K)^{-1})$$

Moreover, if $\{K_n\}_{n \in \mathbb{N}}$ is a sequence of selfadjoint bounded operators such $\sup_n \|K_n\| < \inf \sigma(T)$ and $\lim_{n \rightarrow \infty} \|K_n\| = 0$, then

$$\lim_{n \rightarrow \infty} \det(I + \overline{(T + K_n)^{-1}C}) = \det(I + T^{-1}C) \quad (\text{A.4})$$

and

$$\lim_{n \rightarrow \infty} \det(I + C(T + K_n)^{-1}) = \det(I + CT^{-1}). \quad (\text{A.5})$$

For any $\varepsilon \in (0, \inf \sigma(T))$ there is a Hilbert-Schmidt operator K_ε such that $\|K_\varepsilon\|_{\mathfrak{S}_2} < \varepsilon$ and $T + K_\varepsilon$ has pure point spectrum. Let λ_n , $n \in \mathbb{N}$, be an enumeration of the eigenvalues of $T + K_\varepsilon$ where without loss of generality we may assume that the eigenvalues are simple. Let P_m the orthogonal projection onto the subspace which is spanned by the first m -eigenspaces. Obviously, we have $s - \lim_{m \rightarrow \infty} P_m = I$ and $(T + K_\varepsilon)P_m = P_m(T + K_\varepsilon)$, $m \in \mathbb{N}$. We find

$$\begin{aligned} s - \lim_{n \rightarrow \infty} \det(I + (T + K_\varepsilon)^{-1}P_m C) &= \\ s - \lim_{n \rightarrow \infty} \det(I + P_m C(T + K_\varepsilon)^{-1}) &= \det(I + C(T + K)^{-1}). \end{aligned}$$

and

$$\begin{aligned} s - \lim_{n \rightarrow \infty} \det(I + (T + K_\varepsilon)^{-1}P_m C) &= \\ s - \lim_{n \rightarrow \infty} \det(I + P_m \overline{(T + K_\varepsilon)^{-1}C}) &= \det(I + \overline{(T + K_\varepsilon)^{-1}C}) \end{aligned}$$

which yields

$$\det(I + \overline{(T + K_\varepsilon)^{-1}C}) = \det(I + C(T + K_\varepsilon)^{-1}).$$

Choosing a sequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$, $\varepsilon_n > 0$, which tends to zero as $n \rightarrow \infty$ we get

$$\det(I + \overline{(T + K_{\varepsilon_n})^{-1}C}) = \det(I + C(T + K_{\varepsilon_n})^{-1}).$$

for each $n \in \mathbb{N}$. Taking into account (A.4) and (A.5) we verify (A.3). \square

Corollary A.4. *Let T be a densely defined closed symmetric operator such that $0 \in \rho(T)$. Further, let C be a linear operator such that $\text{dom}(C) \supseteq \text{ran}(T)$. If $\overline{T^{-1}C} \in \mathfrak{S}_1(\mathfrak{H})$ and $CT^{-1} \in \mathfrak{S}_1(\mathfrak{H})$, then*

$$\det(I + \overline{T^{-1}C}) = \det(I + CT^{-1}). \quad (\text{A.6})$$

Proof. By the polar decomposition we have $T^* = U|T^*|$ where U is unitary. Hence, $T^{-1} = U|T^*|^{-1}$. Since $U^*T^{-1}C = |T^*|^{-1}C$ one has $\overline{|T^*|^{-1}C} \in \mathfrak{S}_1(\mathfrak{H})$. From $CT^{-1} = \tilde{C}|T^*|^{-1} \in \mathfrak{S}_1(\mathfrak{H})$ where $\tilde{C} = CU$. Obviously, we find

$$\det(I + T^{-1}C) = \det(I + U|T^*|^{-1}C) = \det(I + |T^*|^{-1}\tilde{C})$$

and

$$\det(I + CT^{-1}) = \det(I + \tilde{C}|T^*|^{-1}).$$

Applying Lemma A.3 we get $\det(I + |T^*|^{-1}\tilde{C}) = \det(I + \tilde{C}|T^*|^{-1})$ which yields (A.6) \square

B Perturbation determinant and their properties

Let us summarize some important properties of perturbation determinants for additive perturbations $\Delta_{H'/H}(\cdot)$, cf. [65, Section 8.1] and [9, 32].

Definition B.1 ([32, Chapter I.2]).

(i) A vector $\varphi \in \mathfrak{H} \setminus \{0\}$ is called a root vector of a closed operator $T \in \mathcal{C}(\mathfrak{H})$ corresponding to its eigenvalue $\lambda_0 \in \sigma_p(T)$ if there exists $n \in \mathbb{N}$ such that $(T - \lambda_0)^n \varphi = 0$. The closure of the set $\mathfrak{L}'_{\lambda_0}(T)$ of all root vectors of T corresponding to λ_0 is called the root subspace.

$$\mathfrak{L}_{\lambda_0}(T) = \overline{\mathfrak{L}'_{\lambda_0}(T)}, \quad \mathfrak{L}'_{\lambda_0}(T) = \{f \in \mathfrak{H} : (T - \lambda_0)^n f = 0 \text{ for some } n \in \mathbb{N}\}.$$

(ii) The dimension $m_0 = m_{\lambda_0}(A) = \dim \mathfrak{L}_{\lambda_0}(A)$ is called the algebraic multiplicity of λ_0 .

(iii) An eigenvalue $\lambda_0 \in \sigma_p(T)$ is called a normal eigenvalue of T if it is isolated and its algebraic multiplicity $m_{\lambda_0}(T)$ is finite, $m_{\lambda_0}(T) < \infty$.

If $m_0 < \infty$, then $\mathfrak{L}'_{\lambda_0}(A)$ is closed and $= \mathfrak{L}_{\lambda_0}(A)$ is closed. An isolated eigenvalue $\lambda_0 \in \sigma_p(T)$ is a normal one if and only if

$$m_0 = \dim P_{\lambda_0} < \infty, \quad P_{\lambda_0} = -\frac{1}{2\pi i} \int_{|z - \lambda_0| = \delta} R_T(z) dz.$$

We set $m_{\lambda_0}(T) = 0$, if z_0 is regular, i.e. $z_0 \in \rho(T)$.

Further, if the function $f(\cdot)$ is analytic in a punctured neighborhood of $z_0 \in \mathbb{C}$ and z_0 is not an essential singularity of it, then the order $\text{ord}(f(z_0))$ of $f(\cdot)$ at $z_0 \in \mathbb{C}$ is the integer $k \in \mathbb{Z}$ in the representation $f(z) = (z - z_0)^k g(z)$ where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$, [32, Chapter IV.3].

If H' and H are densely defined closed operators in the Hilbert space \mathfrak{H} such that $\{H', H\} \in \mathfrak{D}$, then the perturbation determinant defined by (1.1) has the following properties:

1. If $H', H \in \mathfrak{S}_1(\mathfrak{H})$, then $\{H', H\} \in \mathfrak{D}$ and

$$\Delta_{H'/H}(z) = \frac{\det(I - z^{-1}H')}{\det(I - z^{-1}H)}, \quad z \in \rho(H) \setminus \{0\}. \quad (\text{B.1})$$

2. If $\{H'', H'\} \in \mathfrak{D}$ and $\{H', H\} \in \mathfrak{D}$, then $\{H'', H\} \in \mathfrak{D}$ and the following chain rule holds

$$\Delta_{H'', H'}(z) \Delta_{H', H}(z) = \Delta_{H'', H}(z), \quad z \in \rho(H') \cap \rho(H). \quad (\text{B.2})$$

3. If $\{H', H\} \in \mathfrak{D}$, then $\{H, H'\} \in \mathfrak{D}$ and $\Delta_{H'/H}(z) \Delta_{H/H'}(z) = 1$ for $z \in \rho(H') \cap \rho(H)$.
4. If $\{H', H\} \in \mathfrak{D}$ and z is either a common regular point or a normal eigenvalue of both H' and H of algebraic multiplicities $m_z(H')$ and $m_z(H)$, then $\text{ord}(\Delta_{H'/H}(z)) = m_z(H') - m_z(H)$.
5. If $\{H', H\} \in \mathfrak{D}$, then

$$\begin{aligned} \frac{1}{\Delta_{H'/H}(z)} \frac{d}{dz} \Delta_{H'/H}(z)^{-1} = \\ \text{tr}((H - z)^{-1} - (H' - z)^{-1}), \quad z \in \rho(H') \cap \rho(H). \end{aligned} \quad (\text{B.3})$$

6. If $\{H', H\} \in \mathfrak{D}$ and $\{H'^*, H^*\} \in \mathfrak{D}$, then $\Delta_{H'^*/H^*}(z) = \overline{\Delta_{H'/H}(\bar{z})}$ for $z \in \rho(H^*)$.

7. If $\{H', H\} \in \mathfrak{D}$, then the following identity holds

$$\frac{\Delta_{H'/H}(z)}{\Delta_{H'/H}(\zeta)} = \det(I + (z - \zeta)(H' - \zeta)^{-1}V(H - z)^{-1}),$$

$$z \in \rho(H) \text{ and } \zeta \in \rho(H') \cap \rho(H).$$

C Logarithm

In the following we need the definition of the logarithm $\log(z)$ of a complex number $z \in \mathbb{C}$. We shall define the logarithm by

$$\log(z) := -i \int_0^\infty ((z + i\lambda)^{-1} - (1 + i\lambda)^{-1}) d\lambda, \quad z \in \mathbb{C}_+ \setminus -i\mathbb{R}_+, \quad (\text{C.1})$$

with a cut along the negative imaginary semi-axis. One proves that

$$\log(e^z) = z, \quad e^z \in \mathbb{C}_+ \setminus -i\mathbb{R}_+,$$

which yields

$$e^{\log(z)} = z, \quad z \in \mathbb{C}_+ \setminus -i\mathbb{R}_+.$$

Let $f(\cdot)$ and $g(\cdot)$ be holomorphic functions in a domain Ω satisfying $f(z) \neq 0$ and $f(z) = e^{g(z)}$. Then for a neighborhood \mathcal{O} of a fixed point $z_0 \in \Omega$ such that $f(z_0)$ does not belong to the negative imaginary semi-axis one has

$$\log(f(z)) = g(z) + 2n\pi i, \quad z \in \mathcal{O}, \quad n \in \mathbb{Z}.$$

By analytical continuation this equality can be extended to the whole Ω . Using definition (C.1) we find

$$\frac{d}{dz} \log(f(z)) = \frac{1}{f(z)} \frac{d}{dz} f(z), \quad z \in \Omega.$$

Furthermore, we need the definition of the logarithm of a dissipative operator G given in [26]. Let a G be a bounded dissipative operator such that $0 \in \rho(G)$. Then we set

$$\log(G) := -i \int_0^\infty ((G + i\lambda)^{-1} - (1 + i\lambda)^{-1}) d\lambda \quad (\text{C.2})$$

where the integral is understood in the operator norm. One proves that $e^{\log(G)} = G$, cf. [26, Lemma 2.5(e)]. Moreover, from [26, Lemma 2.6] we find $0 \leq \text{Im}(\log(G)) \leq \pi I$. If $G \in \mathfrak{S}_1(\mathfrak{H})$ and dissipative, then $\log(I + G) \in \mathfrak{S}_1(\mathfrak{H})$. Moreover, one gets

$$\det(I + G) = e^{\text{tr}(\log(I+G))}, \quad G \in \mathfrak{S}_1(\mathfrak{H}). \quad (\text{C.3})$$

D Holomorphic functions in \mathbb{C}_+

A holomorphic function $F_+(\cdot)$ in \mathbb{C}_+ belongs to $F_+(\cdot) \in H^\infty(\mathbb{C}_+)$ if $\sup_{z \in \mathbb{C}_+} |F_+(z)| < \infty$. Let $\{z_k^+\}_{k \in \mathbb{N}} (\subset \mathbb{C}_+)$ be the set of its zeros and m_k the corresponding multiplicities. It is well known (see for instance [38, Section VI C]) that the set of zeros satisfy the condition

$$\sum_k \frac{m_k \text{Im}(z_k^+)}{1 + |z_k^+|^2} < \infty. \quad (\text{D.1})$$

If the sequence $\{\alpha_k^+\}_{k \in \mathbb{N}} \subset \mathbb{R}$ is chosen such that $e^{i\alpha_k^+}(i - z_k^+)(i - \bar{z}_k^+)^{-1} \geq 0$, $k \in \mathbb{N}$, then the corresponding Blaschke product

$$\mathcal{B}_+(z) = \prod_k (b_{z_k})^{m_k} := \prod_k \left(e^{i\alpha_k^+} \frac{z - z_k^+}{z - \bar{z}_k^+} \right)^{m_k}, \quad z \in \mathbb{C}_+, \quad (\text{D.2})$$

converges uniformly on compact subsets of \mathbb{C}_+ .

Moreover, $F_+(\cdot)$ admits (see [38, Section VI C]) the following representation

$$F_+(z) = \varkappa_+ \mathcal{B}_+(z) \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\mu_+(t) \right\} e^{i\alpha_+ z}, \quad (\text{D.3})$$

$z \in \mathbb{C}_+$, where $\varkappa_+ \in \mathbb{T}$, $\alpha_+ \geq 0$ and $\mu_+(\cdot)$ is a non-decreasing function on \mathbb{R} generating a non-negative Borel measure and satisfying

$$\int_{\mathbb{R}} \frac{1}{1+t^2} d\mu_+(t) < \infty. \quad (\text{D.4})$$

If $F_+(z)$ has no zeros in \mathbb{C}_+ , the Blaschke product $\mathcal{B}_+(\cdot)$ in (D.3) is missing. Let $\mu_+ = \mu_+^s + \mu_+^{ac}$ be the Lebesgue decomposition of μ_+ , where μ_+^s and μ_+^{ac} are the singular and the absolutely continuous measures, respectively. Setting

$$\begin{aligned} I_{F_+}(z) &:= \mathcal{B}_+(z) S_{F_+}(z) e^{i\alpha_+ z}, \\ S_{F_+}(z) &:= \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu_+^s(t) \right\}, \end{aligned} \quad (\text{D.5})$$

where $\alpha_+ \geq 0$ and

$$\mathcal{O}_{F_+}(z) := \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu_+^{ac}(t) \right\}, \quad z \in \mathbb{C}_+,$$

one gets the unique factorization $F_+(z) = \varkappa_+ I_{F_+}(z) \mathcal{O}_{F_+}(z)$, $z \in \mathbb{C}_+$, where $\varkappa_+ \in \mathbb{T}$ and $I_{F_+}(z)$ and $\mathcal{O}_{F_+}(z)$ are the inner and the outer factors, respectively. Note, that $|I_{F_+}(t+i0)| = 1$, $|\mathcal{O}_{F_+}(t+i0)| = |F_+(t+i0)|$ for a.e. $t \in \mathbb{R}$ and $d\mu_+^{ac}(t) = -\ln(|F_+(t+i0)|)dt$. Note, that $F_+(\cdot)$ is an outer function in \mathbb{C}_+ , if and only if it admits the representation

$$F_+(z) = \varkappa_+ \exp \left\{ -\frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \ln(|F_+(t+i0)|) dt \right\}, \quad z \in \mathbb{C}_+. \quad (\text{D.6})$$

Clearly, $F_+(i)$ is real, if and only if $\varkappa_+ = 1$.

A holomorphic function F belongs to the Smirnov class $\mathcal{N}^+(\mathbb{C}_+)$ if it admits the representation $F = F_+/G$ where $F_+, G \in H^\infty(\mathbb{C}_+)$ and G is an outer function. Any function $F \in \mathcal{N}^+(\mathbb{C}_+)$ admits the representation

$$\begin{aligned} F(z) &= \varkappa B_+(z) \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-i} - \frac{t}{1+t^2} \right) d\mu_+^s(t) \right\} \times \\ &\exp \left\{ -\frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-i} - \frac{t}{1+t^2} \right) h(t) dt \right\} e^{i\alpha z}, \quad z \in \mathbb{C}_+, \end{aligned} \quad (\text{D.7})$$

where $\varkappa \in \mathbb{T}$, $\alpha \geq 0$, $\mu_+^s(\cdot)$ is a non-negative Borel measure, which is singular with respect to the Lebesgue measure, and $h \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$. Using (D.7) one easily verifies that the $H^\infty(\mathbb{C}_+)$ -functions F_+ and G can be chosen contractive. Indeed, let $\eta(t) := \max\{h(t), 0\} \geq 0$, $t \in \mathbb{R}$, and $k(t) := \eta(t) - h(t) \geq 0$, $t \in \mathbb{R}$. Notice that $h(t) = \eta(t) - k(t)$, $t \in \mathbb{R}$. Setting $d\mu_+(\cdot) := d\mu_+^s(\cdot) + k(\cdot)dt$,

$$F_+(z) := \varkappa B_+(z) \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-i} - \frac{t}{1+t^2} \right) d\mu_+(t) \right\} e^{i\alpha z}, \quad z \in \mathbb{C}_+,$$

and

$$G(z) := \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-i} - \frac{t}{1+t^2} \right) \eta(t) dt \right\} \quad z \in \mathbb{C}_+,$$

we define contractive analytic functions, where G is an outer function, such that $F = F_+/G$. Summing up we have proved the following lemma.

Lemma D.1. *If $F \in \mathcal{N}^+(\mathbb{C}_+)$, then there exists a non-negative Borel measure $\mu_+(\cdot)$ satisfying $\int_{\mathbb{R}} \frac{d\mu_+(t)}{1+t^2} dt < \infty$ and a non-negative function $\eta(\cdot) \in L^1(\mathbb{R}, \frac{1}{1+t^2} dt)$ as well as constants $\varkappa \in \mathbb{T}$ and $\alpha \geq 0$ such that the representation*

$$F(z) = \varkappa B_+(z) \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{t-i} - \frac{t}{1+t^2} \right) d\mu(t) \right\} \quad (\text{D.8})$$

holds where $d\mu(\cdot) = d\mu_+(\cdot) - \eta(\cdot)dt$.

E On $H^1(\mathbb{D})$ functions

Let $\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$. By $\ln(\cdot)$ we denote a branch of the logarithm such that $\ln(z) \in \mathbb{R}$ for $z \in \mathbb{R}_+$ and $\text{Im}(\ln(z)) \in (-\pi/2, \pi/2)$ for $\text{Re}(z) > 0$.

Lemma E.1. *Let $H(w)$ be a holomorphic function in \mathbb{D} such that $\text{Re}(H(w)) \geq 0$ for $w \in \mathbb{D}$. Let $G(w) := \ln(1 + H(w))$ for $w \in \mathbb{D}$. Then $G(w) \in H^1(\mathbb{D})$ and the following estimate holds*

$$0 \leq \int_{-\pi}^{\pi} \text{Re}(G(e^{i\theta})) d\theta \leq 2\pi |H(0)|. \quad (\text{E.1})$$

Proof. Obviously we have $|\text{Im}(G(w))| \leq \pi/2$, $w \in \mathbb{D}$. Furthermore, we have

$$G(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(re^{i\theta}) d\theta, \quad r \in (0, 1),$$

which yields

$$2\pi \text{Re}(G(0)) = \int_{-\pi}^{\pi} \text{Re}(G(re^{i\theta})) d\theta.$$

Since $\text{Re}(G(re^{i\theta})) \geq 0$ we obtain

$$\|G\|_{H^1} \leq 2\pi \text{Re}(G(0)) + \pi^2$$

which yields $G \in H^1(\mathbb{D})$. In particular, we have

$$\|G_R\|_{L^1} = 2\pi G_R(0), \quad G_R(w) = \text{Re}(G(w)), \quad w \in \mathbb{D}. \quad (\text{E.2})$$

Using the estimate $\text{Re}(G(0)) = \ln(|1 + H(0)|) \leq |H(0)|$ we arrive at (E.1). \square

The result can be carried over to upper half-plane.

Corollary E.2. *Let $h(z)$, $z \in \mathbb{C}_+$, be a holomorphic function such that $\text{Re}(h(z)) \geq 0$ for $z \in \mathbb{C}_+$. Let $g(z) := \ln(1 + h(z))$ for $z \in \mathbb{C}_+$. Then the following estimate*

$$\int_{\mathbb{R}} |g(x + i0)| \frac{dx}{1+x^2} \leq 2\pi |h(i)| \quad (\text{E.3})$$

is valid where $g(x + i0) := \lim_{y \downarrow 0} g(x + iy)$.

Proof. We set

$$H(w) := h\left(i\frac{1+w}{1-w}\right)$$

and

$$G(w) := \ln(1 + H(w)) = \ln\left(1 + h\left(i\frac{1+w}{1-w}\right)\right) = g\left(i\frac{1+w}{1-w}\right).$$

Since

$$\int_{-\pi}^{\pi} |G(e^{i\theta})| d\theta = \int_{\mathbb{R}} |g(x + i0)| \frac{dx}{1+x^2}$$

and $h(i) = H(0)$ we obtain (E.3) from (E.1). \square

F Riesz-Dunford functional calculus

Let T be a densely defined closed operator. We say the function Φ belongs to the class $\mathcal{F}(T)$ if there is a simple closed curve Γ in \mathbb{C} which does not intersect the real axis such that

- (i) its open exterior domain $\Omega_{\Gamma}^{\text{ext}}$ contains the spectra of T and
- (ii) there is a neighborhood \mathcal{O} of the closed set $\overline{\Omega_{\Gamma}^{\text{ext}}}$ such that Φ is holomorphic in \mathcal{O} including infinity.

We note that if $\rho(T)$ is not empty, then the class $\mathcal{F}(T)$ is not an empty too. In this case one defines $\Phi(T)$ by

$$\Phi(T) := \Phi(\infty)I + \frac{1}{2\pi i} \oint_{\Gamma} \Phi(z)(T - z)^{-1} dz \quad (\text{F.1})$$

where the integral \oint_{Γ} is taken in mathematical positive sense with respect to open inner domain $\Omega_{\Gamma}^{\text{in}}$, see [21, Section VII.9]. We note that

$$\Phi(\xi) = \Phi(\infty) - \frac{1}{2\pi i} \oint_{\Gamma} \frac{\Phi(z)}{z - \xi} dz, \quad \xi \in \Omega_{\Gamma}^{\text{ext}}, \quad (\text{F.2})$$

Since

$$\oint_{\Gamma} |\Phi(z)| |dz| < \infty \quad (\text{F.3})$$

the integral $\oint_{\Gamma} \Phi(z) dz$ is well-defined. Hence the residuum $\text{res}_{\infty}(\Phi)$, $\Phi \in \mathcal{F}(T)$, is well defined by

$$\text{res}_{\infty}(\Phi) := -\frac{1}{2\pi i} \oint_{\Gamma} \Phi(z) dz \quad (\text{F.4})$$

Since the curve Γ does not intersect the real axis, we get from (F.2) and (F.3) that $\sup_{t \in \mathbb{R}} (1+t^2)|\Phi'(t)| < \infty$ for $\Phi \in \mathcal{F}(T)$. Therefore, if ν is a complex-valued Borel measure satisfying $\int_{\mathbb{R}} \frac{1}{1+t^2} |d\nu(t)| < \infty$, then

$$\int_{\mathbb{R}} |\Phi'(t)| |d\nu(t)| < \infty$$

which guarantees the existence of the integral $\int_{\mathbb{R}} \Phi'(t) d\nu(t)$ and

$$\int_{\mathbb{R}} \Phi'(t) d\nu(t) = -\frac{1}{2\pi i} \oint_{\Gamma} \Phi(z) \left(\int_{\mathbb{R}} \frac{1}{(t-z)^2} d\nu(t) \right) dz \quad (\text{F.5})$$

for $\Phi \in \mathcal{F}(T)$ provided ν satisfies $\int_{\mathbb{R}} \frac{1}{1+t^2} |d\nu(t)| < \infty$.

Let T and T' be two densely defined closed operators. We set $\mathcal{F}(T, T') := \mathcal{F}(T) \cap \mathcal{F}(T')$, that is, there is a simple closed curve Γ such that $\Omega_{\Gamma}^{\text{ext}}$ contains the spectra of both T and T' . If $\rho(T) \cap \rho(T') \neq \emptyset$, then $\mathcal{F}(T, T') \neq \emptyset$.

Lemma F.1. *Let T and T' be two densely defined closed operators in \mathfrak{H} such that $\rho(T) \cap \rho(T') \neq \emptyset$. If the condition $(T' - \xi)^{-1} - (T - \xi)^{-1} \in \mathfrak{S}_1(\mathfrak{H})$ for some $\xi \in \rho(T) \cap \rho(T')$, then $\Phi(T') - \Phi(T) \in \mathfrak{S}_1(\mathfrak{H})$ for $\Phi \in \mathcal{F}(T, T')$.*

Proof. Obviously we have

$$\Phi(T') - \Phi(T) = \frac{1}{2\pi i} \oint_{\Gamma} \Phi(z) ((T' - z)^{-1} - (T - z)^{-1}) dz$$

which yields the estimate

$$\|\Phi(T') - \Phi(T)\|_{\mathfrak{S}_1} \leq \sup_{z \in \Gamma} \|(T' - z)^{-1} - (T - z)^{-1}\|_{\mathfrak{S}_1} \frac{1}{2\pi} \oint_{\Gamma} |\Phi(z)| |dz| < \infty$$

which proves the assertion. \square

G Root vectors

Let T be an unbounded operator in \mathfrak{H} . If $0 \neq \psi \in \ker((T - \lambda)^n)$ for some $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$, then ψ is called a root vector of T belonging to λ . The point λ is necessarily an eigenvalue of T . The set of all root vectors belonging to λ is denoted by $\mathcal{V}_T(\lambda)$. The set $\mathcal{V}_T = \bigcup_{\lambda \in \sigma(T)} \mathcal{V}_T(\lambda)$ is called the root vector system of T . If \mathcal{V}_T is a total set, then the root vector system is called complete.

Lemma G.1. *Let T be a densely defined closed operator such that $\xi \in \rho(T)$. The root vector system of T is complete if and only if the root vector system of $R := (T - \xi)^{-1}$ is complete.*

Proof. Using

$$(R - \mu)^n \psi = (-1)^n \mu^n (T - \xi)^{-n} (T - \lambda)^n \psi, \quad \mu := \frac{1}{\lambda - \xi}, \quad \psi \in \mathfrak{H}, \quad (\text{G.1})$$

we get that $\psi \in \mathcal{V}_T(\lambda)$ yields $\psi \in \mathcal{V}_R(\mu)$. Conversely, if $\psi \in \mathcal{V}_R(\mu)$, then

$$0 = (R - \mu)^n \psi = \sum_{k=0}^n \binom{n}{k} \mu^{n-k} R^k \psi = \mu^n \psi + \sum_{k=1}^n \binom{n}{k} \mu^{n-k} R^k \psi$$

which implies

$$\psi = -R \sum_{k=0}^{n-1} \binom{n}{k+1} \mu^{-(k+1)} R^k \psi.$$

Hence, $\psi \in \text{dom}(T)$, and

$$\psi = -R \sum_{k=0}^{n-1} \binom{n}{k+1} \mu^{-(k+1)} R^{k+1} (T - \lambda) \psi.$$

This yields

$$\psi = -R^2 \sum_{k=0}^{n-1} \binom{n}{k+1} \mu^{-(k+1)} R^k (T - \lambda) \psi.$$

and $\psi \in \text{dom}(T^2)$. If we proceed further in this way we finally get $\psi \in \text{dom}((A - \lambda)^n)$. Applying again formula (G.1) we obtain $\psi \in \mathcal{V}_T(\lambda_0)$. \square

H The class C_0

Let us briefly recall some basic concepts and facts on contractions following [61]. In [61] Nagy and Foias using theory of dilations have extended the Riesz-Dunford functional calculus for a contraction T to the class $H_T^\infty(\mathbb{D})$ (see [61, Section 3.2] for precise definitions). If a contraction T is completely non-unitary, then $H_T^\infty(\mathbb{D}) = H^\infty(\mathbb{D})$ is just the Hardy class in the unit disc \mathbb{D} . The extended functional calculus makes it possible to introduce concepts of C_0 -contractions and minimal annihilation function.

Definition H.1 ([61]).

- (i) A contraction T in \mathfrak{H} is put in the class C_0 . (C_0) if $s - \lim_{n \rightarrow \infty} T^n = 0$ ($s - \lim_{n \rightarrow \infty} T^{*n} = 0$). It is set $C_{00} := C_0 \cap C_{0*}$.
- (ii) It is said that a completely non-unitary operator T belongs to the class C_0 if there exists a function $u(\cdot) \in H^\infty(\mathbb{D}) \setminus \{0\}$ such that $u(T) = 0$. The function $u(\cdot)$ is called an annihilation function for T .
- (iii) An annihilation function $u_0(\cdot)$ is called minimal if it is a divisor in $H^\infty(\mathbb{D})$ of any other annihilation function $u(\cdot)$ for T .

It is well known that $C_0 \subset C_{00}$. Moreover, it is known [61, Proposition 3.4.4] that for any $T \in C_0$ the minimal function exists and is unique up to a multiplicative constant. The minimal function is denoted by $m_T(\cdot)$. It is always an inner one.

Alongside a m -dissipative operator D in \mathfrak{H} we consider its Cayley transform

$$T := T_D := (D - i)(D + i)^{-1} = I - 2i(D + i)^{-1}.$$

Clearly, T_D is a contraction. Moreover, T_D is completely non-unitary if and only if D is completely non-selfadjoint.

For any function $v(\cdot) \in H^\infty(\mathbb{C}_+)$ and any completely non-selfadjoint m -dissipative operator D we set $v(D) := \tilde{v}(T_D)$ where

$$H^\infty(\mathbb{D}) \ni \tilde{v}(\zeta) := v\left(i\frac{1+\zeta}{1-\zeta}\right), \quad \zeta \in \mathbb{D}.$$

We say the m -dissipative operator D belongs to the class C_0 . ($C_{\cdot 0}$, C_0) if T_D belongs to C_0 . (resp. $C_{\cdot 0}$, C_0). In other words, $D \in C_0$ if it is completely non-selfadjoint and there is a function $v(\cdot) \in H^\infty(\mathbb{C}_+)$ such that $v(D) = \tilde{v}(T_D) = 0$. Clearly, there always exists a minimal function $m_D(\cdot) \in H^\infty(\mathbb{C}_+)$ which is an inner function.

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